BANDWIDTH SELECTION FOR KERNEL DENSITY ESTIMATION OF FAT-TAILED AND SKEWED DISTRIBUTIONS

TECHNICAL APPENDIX

ABSTRACT. This Technical Appendix includes various mathematical derivations that relate to the results shown in the paper. It is available from the authors upon request.

JEL Classification:C1 (General), C13 (Estimation), C14 (Semiparametric and nonparametric methods).

1. Standardized Roughness

Let $f(x)$ be the density of a distribution that is followed by random variable X, supported in $(-\infty, \infty)$. Let Z be a scaled version of X, $Z = sX,\ s > 0$. Then

$$f_z(z) = \frac{1}{s} f_x(z/s), \quad f_z^{(1)}(z) = \frac{1}{s^2} f_x^{(1)}(z/s), \quad f_z^{(2)}(z) = \frac{1}{s^3} f_x^{(2)}(z/s).$$

Then,

$$R[f_z^{(2)}] = \int_{-\infty}^{\infty} \left[f_z^{(2)}(z)\right]^2 dz = \frac{1}{s^6} \int_{-\infty}^{\infty} \left[f_x^{(2)}(z/s)\right]^2 dz$$

Applying the change of variables $t = z/s \implies dz = sdt$ we get

$$R[f_z^{(2)}] = \frac{1}{s^5} \int_{-\infty}^{\infty} \left[f_x^{(2)}(t)\right]^2 dt = \frac{1}{s^5} R[f_x^{(2)}]$$

If we want Z to have unitary variance, we must set $s = 1/\sigma_x$ and so we arrive at

$$R[f_1^{(2)}] = \sigma_x^5 R[f_x^{(2)}] \implies R[f_x^{(2)}] = \sigma_x^{-5} R[f_1^{(2)}].$$

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2. Roughness of Beta(4,4) distribution

. The density of the Beta(4, 4) distribution is

$$f(x) = \frac{x^3(1-x^3)}{\mathrm{B}(4,4)} = \frac{\Gamma(8)}{\Gamma(4)\Gamma(4)}x^3(1-x^3) = \frac{7!}{3!3!}x^3(1-x^3) = 140x^3(1-x^3).$$

Derivatives are

$$\begin{aligned}
f^{(1)} &= 140[3x^2(1-x)^3 - 3x^3(1-x)^2] = 420[x^2(1-x)^3 - x^3(1-x)^2] \\
f^{(2)} &= 420[2x(1-x)^3 - 3x^2(1-x)^2 - 3x^2(1-x)^2 + 2x^3(1-x)] \\
&= 420[2x(1-x)^3 - 6x^2(1-x)^2 + 2x^3(1-x)], \\
\left[f^{(2)}\right]^2 &= (420)^2\left\{[2x(1-x)^3 - 6x^2(1-x)^2]^2 + 4x^6(1-x)^2 + 4x^3(1-x)[2x(1-x)^3 - 6x^2(1-x)^2]\right\} \\
&= (420)^2\Big[4x^2(1-x)^6 - 24x^3(1-x)^5 + 36x^4(1-x)^4 \\
&\qquad + 4x^6(1-x)^2 + 8x^4(1-x)^4 - 24x^5(1-x)^3\Big]
\end{aligned}$$

We have everywhere the kernel of the Beta density, so

$$\begin{aligned}
R[f^{(2)}] = \int_0^1 \left[f^{(2)}(x)\right]^2 dx &= (420)^2\Big[4\mathrm{B}(3,7) - 24\mathrm{B}(4,6) + 44\mathrm{B}(5,5) + 4\mathrm{B}(7,3) - 24\mathrm{B}(6,4)\Big] \\
&= (420)^2\Big[8\mathrm{B}(3,7) - 48\mathrm{B}(4,6) + 44\mathrm{B}(5,5)\Big] \\
&= (420)^2\left[8\frac{2!6!}{9!} - 48\frac{3!5!}{9!} + 44\frac{4!4!}{9!}\right] = \frac{(420)^2}{9!}\cdot 2304 = 1120.
\end{aligned}$$

This is the roughness of the Beta $(4, 4)$ that has standard deviation

$$SD = \left(\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}\right)^{1/2} \longrightarrow_{(4,4)} = \left(\frac{16}{64\cdot 9}\right)^{1/2} \approx 0.1667.$$

The small value of the standard deviation explains why the roughness is so high. The roughness of the distribution scaled to have unitary variance is

$$R[f_1^{(2)}] = (0.1667)^5 \cdot 1120 \approx 0.144.$$

3. Alternative expression for distribution moments

For a distribution with zero mean and unitary variance, we have $\gamma_2 + 3 = \mu_4$ and the 4th central moment is

$$\mu_4 \int_{-\infty}^{\infty} t^4 f_1(t)dt = \frac{1}{5}\int_{-\infty}^{\infty} \frac{dt^5}{dt} f_1(t)dt.$$

Applying twice integration by parts we have

$$\begin{aligned}\mu_4 &= \frac{1}{5}t^5 f_1(t)\Big|_{-\infty}^{\infty} - \frac{1}{5}\int_{-\infty}^{\infty} t^5 f_1^{(1)}(t)dt = 0 - \frac{1}{30}\int_{-\infty}^{\infty} \frac{dt^6}{dt} f_1^{(1)}(t)dt \\ &= -\frac{1}{30}t^6 f_1^{(1)}\Big|_{-\infty}^{\infty} + \frac{1}{30}\int_{-\infty}^{\infty} t^6 f_1^{(2)}(t)dt = \frac{1}{30}\int_{-\infty}^{\infty} t^6 f_1^{(2)}(t)dt.\end{aligned}$$

4. Approximating the Roughness of a distribution

The general expression for a second-order Gram-Charlier type A series expansion of the density of a distribution is

$$f(x) \approx \phi\left(\frac{x-\mu}{\sigma}\right)\left[1 + \frac{\gamma_1}{3!}He_3\left(\frac{x-\mu}{\sigma}\right) + \frac{\gamma_2}{4!}He_4\left(\frac{x-\mu}{\sigma}\right)\right],$$

where μ is the mean, σ is the standard deviation, γ_1 and γ_2 are the skewness and excess kurtosis coefficients respectively, and He_n are Hermite polynomials (as used in probability theory).

We standardize the distribution, setting $\sigma = 1$. Also, location does not affect roughness, skewness, or excess kurtosis, so we set $\mu = 0$. Suppresing the "x" argument, we have

$$f \approx \phi \cdot \left[1 + \frac{\gamma_1}{6} \cdot He_3 + \frac{\gamma_2}{24} \cdot He_4\right] = \phi + \frac{\gamma_1}{6} \cdot \phi \cdot He_3 + \frac{\gamma_2}{24} \phi \cdot He_4.$$

It follows that

$$f^{(2)} = \phi^{(2)} + \frac{\gamma_1}{6} \cdot (\phi \cdot He_3)^{(2)} + \frac{\gamma_2}{24} \cdot (\phi \cdot He_4)^{(2)}.$$

We have the relation

$$\phi \cdot He_n = (-1)^n \cdot \phi^{(n)} \implies (\phi \cdot He_n)^{(m)} = (-1)^n \cdot \phi^{(n+m)} = (-1)^m \cdot \phi \cdot He_{n+m}.$$

Applying this to our case ($m = 2$) we get

$$f^{(2)} = \phi^{(2)} + \frac{\gamma_1}{6} \cdot \phi \cdot He_5 + \frac{\gamma_2}{24} \cdot \phi \cdot He_6, \qquad \phi^{(2)} = \phi \cdot He_2.$$

Squaring, we have

$$\begin{aligned}
[f^{(2)}]^2 &= \left[\left(\phi^{(2)} + \frac{\gamma_2}{24} \cdot \phi \cdot He_6\right) + \frac{\gamma_1}{6} \cdot \phi \cdot He_5\right]^2 \\
&= \left(\phi^{(2)} + \frac{\gamma_2}{24} \cdot \phi \cdot He_6\right)^2 + 2 \cdot \left(\phi^{(2)} + \frac{\gamma_2}{24} \cdot \phi \cdot He_6\right) \cdot \frac{\gamma_1}{6} \cdot \phi \cdot He_5 + \frac{\gamma_1^2}{36} \cdot \phi^2 \cdot He_5^2 \\
&= [\phi^{(2)}]^2 + \frac{2}{24}\gamma_2 \cdot \phi^{(2)} \cdot \phi \cdot He_6 + \frac{\gamma_2^2}{(24)^2}\phi^2 \cdot He_6^2 \\
&\qquad + 2 \cdot \left(\phi^{(2)} + \frac{\gamma_2}{24} \cdot \phi \cdot He_6\right) \cdot \frac{\gamma_1}{6} \cdot \phi \cdot He_5 + \frac{\gamma_1^2}{36} \cdot \phi^2 \cdot He_5^2 \\
&= [\phi^{(2)}]^2 + \frac{\gamma_2}{12} \cdot \phi^2 He_2 He_6 + \frac{\gamma_2^2}{(24)^2}\phi^2 He_6^2 \\
&\qquad + \frac{\gamma_1}{3} \cdot \left(\phi^2 He_2 He_5 + \frac{\gamma_2}{24} \cdot \phi^2 \cdot He_6 He_5\right) + \frac{\gamma_1^2}{36} \cdot \phi^2 \cdot He_5^2.
\end{aligned}$$

To obtain the Roughness of f, we will integrate the above expression element by element over $(-\infty, \infty)$. Now note that ϕ^2 is an even function, while the products $He_2 He_5$ and $He_6 He_5$ contain only odd powers of x, which are odd functions. So *their* product with ϕ^2

will be an odd function integrated over $(-\infty, \infty)$. It follows that these integrals will equal zero.

Therefore the Roughness of a distribution with standard deviation equal to 1 is approximated by

$$\begin{aligned} R[f^{(2)}] &\approx R[\phi^{(2)}] + \frac{\gamma_2}{12}\int_{-\infty}^{\infty} \phi^2 \left[He_2 He_6 + \frac{\gamma_2}{48} He_6^2\right] dx + \frac{\gamma_1^2}{36}\int_{-\infty}^{\infty} \phi^2 He_5^2 dx \\ &= R[\phi^{(2)}] + \frac{\gamma_2}{24\pi} \cdot \int_{-\infty}^{\infty} e^{-x^2} \left[He_2 He_6 + \frac{\gamma_2}{48} He_6^2\right] dx \; + \; \frac{\gamma_1^2}{72\pi}\int_{-\infty}^{\infty} e^{-x^2} He_5^2 dx, \end{aligned} \tag{1}$$

where

$$He_2 = x^2 - 1, \qquad He_6 = x^6 - 15x^4 + 45x^2 - 15, \qquad He_5 = x^5 - 10x^3 + 15x.$$

Note that the Hermite polynomials (as used in probability theory) of different order are orthogonal with respect to ϕ and not ϕ^2. Therefore the integral involving $He_2 He_6$ will *not* evaluate to zero.

We move on to obtain an exact expression for the approximation, since the terms inside the integrals ultimately are analyzed in powers of the variable of integration multiplied by the error function and so they have simple closed-form solutions involving values of the Gamma function that in turn are expressed using factorials.

We start with the first integral,

$$\frac{\gamma_2}{24\pi} \cdot \int_{-\infty}^{\infty} e^{-x^2} \left[He_2 He_6 + \frac{\gamma_2}{48} He_6^2\right] dx.$$

We have

$$\begin{aligned} He_2 \cdot He_6 &= x^2 \cdot (x^6 - 15x^4 + 45x^2 - 15) - (x^6 - 15x^4 + 45x^2 - 15) \\ &= x^8 - 15x^6 + 45x^4 - 15x^2 - x^6 + 15x^4 - 45x^2 + 15 \\ &= x^8 - 16x^6 + 60x^4 - 60x^2 + 15. \end{aligned}$$

Also,

$$\begin{aligned}
He_6^2 &= (x^6 - 15x^4 + 45x^2 - 15)\cdot(x^6 - 15x^4 + 45x^2 - 15)\\
&= x^{12} - 15x^{10} + 45x^8 - 15x^6\\
&\quad - 15x^{10} + 15^2x^8 - (15\cdot 45)x^6 + 15^2x^4\\
&\quad + 45x^8 - (45\cdot 15)x^6 + 45^2x^4 - (45\cdot 15)x^2\\
&\quad - 15x^6 + 15^2x^4 - (15\cdot 45)x^2 + 15^2.
\end{aligned}$$

Summing vertically we obtain

$$\begin{aligned}
He_6^2 =& x^{12} - 30x^{10} + (2\cdot 45 + 15^2)x^8\\
&- (2\cdot 15 + 2\cdot 15\cdot 45)x^6 + (2\cdot 15^2 + 45^2)x^4 - (2\cdot 15\cdot 45)x^2 + 15^2\\
\implies He_6^2 =& x^{12} - 30x^{10} + 315x^8 - 1380x^6 + 2475x^4 - 1350x^2 + 225.
\end{aligned}$$

Bringing the two expressions together we have

$$\begin{aligned}
He_2He_6 + \frac{\gamma_2}{48}He_6^2 =& x^8 - 16x^6 + 60x^4 - 60x^2 + 15\\
&+ \frac{\gamma_2}{48}\cdot[x^{12} - 30x^{10} + 315x^8 - 1380x^6 + 2475x^4 - 1350x^2 + 225],
\end{aligned}$$

and arranging per powers of x,

$$\begin{aligned}
He_2He_6 + \frac{\gamma_2}{48}He_6^2 =& \frac{\gamma_2}{48}x^{12} - \frac{30}{48}\gamma_2x^{10} + \left(1 + \frac{315}{48}\gamma_2\right)x^8\\
&- \left(16 + \frac{1380}{48}\gamma_2\right)x^6 + \left(60 + \frac{2475}{48}\gamma_2\right)x^4\\
&- \left(60 + \frac{1350}{48}\gamma_2\right)x^2 + \left(15 + \frac{225}{48}\gamma_2\right).
\end{aligned}$$

If we put all these inside the integral, the full integrand in each case will be an even function,

$$\frac{\gamma_2}{24\pi}\cdot\int_{-\infty}^{\infty}e^{-x^2}\cdot C_{2t}x^{2t}dx=\frac{\gamma_2}{24\pi}\cdot 2C_{2t}\int_{0}^{\infty}e^{-x^2}\cdot x^{2t}dx,$$

where in our case $t = 6, 5, 4, 3, 2, 1, 0$.

Moreover, from Abramowitz and Stegun p. 302 eq. 7.4.4 we have

$$\int_{0}^{\infty}e^{-x^2}\cdot x^{2t}dx=\frac{1}{2}\Gamma(t+1/2).$$

So the integral together with the outside factor will contain components of the form

$$\frac{\gamma_2}{24\pi}\cdot C_{2t}\Gamma(t+1/2)=\frac{\gamma_2}{24\pi}\cdot C_{2t}\frac{(2t)!}{4^t\cdot t!}\sqrt{\pi}=\frac{\gamma_2}{24\sqrt{\pi}}\cdot C_{2t}\frac{(2t)!}{4^t\cdot t!},$$

equaling the sum

$$\frac{\gamma_2}{24\pi}\cdot\int_{-\infty}^{\infty}e^{-x^2}\cdot\left[He_2He_6+\frac{\gamma_2}{48}He_6^2\right]dx=\frac{\gamma_2}{24\sqrt{\pi}}\cdot\sum_{t=0}^{6}C_{2t}\frac{(2t)!}{4^t\cdot t!}.$$

We have, for the elements of the sum:

$$\begin{aligned}x^{12}\to t=6: C_{2t}\frac{(2t)!}{4^t\cdot t!}=\frac{\gamma_2}{48}\cdot\frac{(12)!}{4^6\cdot 6!}&=\gamma_2\cdot\frac{7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12}{6\cdot 8\cdot 4\cdot 4\cdot 4\cdot 4\cdot 4\cdot 4}=\gamma_2\cdot\frac{7\cdot 9\cdot 10\cdot 11\cdot 3}{6\cdot 4\cdot 4\cdot 4\cdot 4\cdot 4}\\&=\gamma_2\cdot\frac{5\cdot 7\cdot 9\cdot 11}{4\cdot 4\cdot 4\cdot 4\cdot 4}=\frac{3465}{1024}\gamma_2.\end{aligned}$$

$$\begin{aligned}x^{10}\to t=5: C_{2t}\frac{(2t)!}{4^t\cdot t!}=-\frac{30}{48}\gamma_2\cdot\frac{(10)!}{4^5\cdot 5!}&=-\gamma_2\cdot\frac{30\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}{48\cdot 4\cdot 4\cdot 4\cdot 4\cdot 4}=-\gamma_2\cdot\frac{30\cdot 7\cdot 9\cdot 10}{4\cdot 4\cdot 4\cdot 4\cdot 4}\\&=-\gamma_2\cdot\frac{7\cdot 9\cdot 75}{4\cdot 4\cdot 4\cdot 4}=-\frac{4725}{256}\gamma_2.\end{aligned}$$

$$\begin{aligned}
x^8 \to t = 4 : C_{2t}\frac{(2t)!}{4^t \cdot t!} &= \left(1 + \frac{315}{48}\gamma_2\right) \cdot \frac{(8)!}{4^4 \cdot 4!} = \left(1 + \frac{315}{48}\gamma_2\right) \cdot \frac{5 \cdot 6 \cdot 7 \cdot 8}{4 \cdot 4 \cdot 4 \cdot 4} \\
&= \left(1 + \frac{315}{48}\gamma_2\right) \cdot \frac{5 \cdot 6 \cdot 7 \cdot 2}{4 \cdot 4 \cdot 4} = \left(1 + \frac{315}{48}\gamma_2\right) \cdot \frac{5 \cdot 7 \cdot 3}{4 \cdot 4} \\
&= \left(1 + \frac{315}{48}\gamma_2\right) \cdot \frac{105}{16} = \frac{105}{16} + \frac{315 \cdot 105}{48 \cdot 16}\gamma_2 \\
&= \frac{105}{16} + \frac{3 \cdot 105^2}{3 \cdot 16^2}\gamma_2 = \frac{105}{16} + \frac{105^2}{16^2}\gamma_2.
\end{aligned}$$

$$\begin{aligned}
x^6 \to t = 3 : C_{2t}\frac{(2t)!}{4^t \cdot t!} &= -\left(16 + \frac{1380}{48}\gamma_2\right) \cdot \frac{(6)!}{4^3 \cdot 3!} = -\left(16 + \frac{1380}{48}\gamma_2\right) \cdot \frac{4 \cdot 5 \cdot 6}{4 \cdot 4 \cdot 4} \\
&= -30 - \frac{1380 \cdot 30}{48 \cdot 16}\gamma_2 = -30 - \frac{69 \cdot 10 \cdot 2 \cdot 5 \cdot 6}{6 \cdot 8 \cdot 2 \cdot 8}\gamma_2 \\
&= -30 - \frac{69 \cdot 10 \cdot 5}{8 \cdot 8}\gamma_2 = -30 - \frac{69 \cdot 25}{32}\gamma_2 = -30 - \frac{1725}{32}\gamma_2.
\end{aligned}$$

$$\begin{aligned}
x^4 \to t = 2 : C_{2t}\frac{(2t)!}{4^t \cdot t!} &= \left(60 + \frac{2475}{48}\gamma_2\right) \cdot \frac{(4)!}{4^2 \cdot 2!} = \left(60 + \frac{2475}{48}\gamma_2\right) \cdot \frac{3}{4} \\
&= 45 + \frac{2475 \cdot 3}{48 \cdot 4}\gamma_2 = 45 + \frac{2475}{16 \cdot 4}\gamma_2 = 45 + \frac{2475}{64}\gamma_2.
\end{aligned}$$

$$x^2 \to t = 1 : C_{2t}\frac{(2t)!}{4^t \cdot t!} = -\left(60 + \frac{1350}{48}\gamma_2\right) \cdot \frac{(2)!}{4 \cdot 1!} = -30 - \frac{1350}{48 \cdot 2}\gamma_2 = -30 - \frac{675}{48}\gamma_2.$$

$$x^0 \to t = 0 : C_{2t}\frac{(2t)!}{4^t \cdot t!} = \left(15 + \frac{225}{48}\gamma_2\right) \cdot \frac{(0)!}{4^0 \cdot 0!} = 15 + \frac{225}{48}\gamma_2.$$

Collecting terms, we have

$$\frac{\gamma_2}{24\pi}\cdot\int_{-\infty}^{\infty}e^{-x^2}\cdot\left[He_2He_6+\frac{\gamma_2}{48}He_6^2\right]dx=$$

$$\frac{\gamma_2}{24\sqrt{\pi}}\cdot\left[\frac{3465}{1024}\gamma_2-\frac{4725}{256}\gamma_2+\frac{105}{16}+\frac{105^2}{16^2}\gamma_2-30-\frac{1725}{32}\gamma_2+45+\frac{2475}{64}\gamma_2-30-\frac{675}{48}\gamma_2+15+\frac{225}{48}\gamma_2\right]$$
$$=\frac{\gamma_2}{24\sqrt{\pi}}\cdot\left[\frac{3465}{1024}\gamma_2-\frac{4725}{256}\gamma_2+\frac{105}{16}+\frac{105^2}{16^2}\gamma_2-\frac{1725}{32}\gamma_2+\frac{2475}{64}\gamma_2-\frac{675}{48}\gamma_2+\frac{225}{48}\gamma_2\right]$$
$$=\frac{\gamma_2}{16\cdot 24\sqrt{\pi}}\cdot\left[\frac{3465}{64}\gamma_2-\frac{4725}{16}\gamma_2+105+\frac{105^2}{16}\gamma_2-\frac{1725}{2}\gamma_2+\frac{2475}{4}\gamma_2-\frac{675}{3}\gamma_2+\frac{225}{3}\gamma_2\right]$$
$$=\frac{105}{384\sqrt{\pi}}\gamma_2+\frac{\gamma_2^2}{384\sqrt{\pi}}\cdot\left[\frac{3465}{64}-\frac{4725}{16}+\frac{105^2}{16}-\frac{1725}{2}+\frac{2475}{4}-\frac{675}{3}+\frac{225}{3}\right].$$

It turns out that the elements inside the brackets cancel out leaving only the first. So we arrive at

$$\frac{\gamma_2}{24\pi}\cdot\int_{-\infty}^{\infty}e^{-x^2}\cdot\left[He_2He_6+\frac{\gamma_2}{48}He_6^2\right]dx=\frac{105}{384\sqrt{\pi}}\gamma_2+\frac{3465}{64\cdot 384\sqrt{\pi}}\gamma_2^2.$$

Factorizing,

$$\frac{\gamma_2}{24\pi}\cdot\int_{-\infty}^{\infty}e^{-x^2}\cdot\left[He_2He_6+\frac{\gamma_2}{48}He_6^2\right]dx=\frac{3\cdot 5\cdot 7}{3\cdot 2^7\sqrt{\pi}}\gamma_2+\frac{3^2\cdot 5\cdot 7\cdot 11}{3\cdot 2^{13}\sqrt{\pi}}\gamma_2^2.$$

(2)
$$=\frac{35}{2^7\sqrt{\pi}}\gamma_2+\frac{1155}{2^{13}\sqrt{\pi}}\gamma_2^2.$$

We turn to the second integral,

$$\frac{\gamma_1^2}{72\pi}\int_{-\infty}^{\infty}e^{-x^2}He_5^2dx.$$

We have

$$\begin{aligned} He_5^2 =& (x^5 - 10x^3 + 15x) \cdot (x^5 - 10x^3 + 15x) \\ =& x^{10} - 10x^8 + 15x^6 \\ & - 10x^8 + 100x^6 - 150x^4 \\ & + 15x^6 - 150x^4 + 225x^2. \end{aligned}$$

Summing vertically we obtain

$$He_5^2 = x^{10} - 20x^8 + 130x^6 - 300x^4 + 225x^2.$$

Following the same procedure as before, we have then that

$$\frac{\gamma_1^2}{72\pi} \int_{-\infty}^{\infty} e^{-x^2} He_5^2 dx = \frac{\gamma_1^2}{72\sqrt{\pi}} \cdot \sum_{t=1}^{5} C_{2t} \frac{(2t)!}{4^t \cdot t!}.$$

$$x^{10} : t = 5, C_{2t} = 1 \to C_{2t} \cdot \frac{(2t)!}{4^t \cdot t!} = 1 \cdot \frac{(10)!}{4^5 \cdot 5!} = \frac{6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4} = \frac{2 \cdot 7 \cdot 9 \cdot 15}{4 \cdot 4 \cdot 4} = \frac{945}{32}.$$

$$\begin{aligned} x^8 : t = 4, C_{2t} = -20 \to C_{2t} \frac{(2t)!}{4^t \cdot t!} =& -20 \cdot \frac{(8)!}{4^4 \cdot 4!} = -(4 \cdot 5) \cdot \frac{5 \cdot 6 \cdot 7 \cdot 8}{4 \cdot 4 \cdot 4 \cdot 4} \\ =& -5 \cdot \frac{5 \cdot 6 \cdot 7 \cdot 2}{4 \cdot 4} = -\frac{5 \cdot 7 \cdot 15 \cdot}{4} = -\frac{525}{4}. \end{aligned}$$

$$x^6 : t = 3, C_{2t} = 130 \to C_{2t} \frac{(2t)!}{4^t \cdot t!} = 130 \cdot \frac{(6)!}{4^3 \cdot 3!} = 2 \cdot 65 \cdot \frac{4 \cdot 5 \cdot 6}{4 \cdot 4 \cdot 4} = \frac{1950}{8}.$$

$$x^4 : t = 2, C_{2t} = -300 \to C_{2t} \frac{(2t)!}{4^t \cdot t!} = -300 \cdot \frac{(4)!}{4^2 \cdot 2!} = -\frac{900}{4}.$$

$$x^2 : t = 1, C_{2t} = 225 \to C_{2t}\frac{(2t)!}{4^t \cdot t!} = 225 \cdot \frac{(2)!}{4 \cdot 1!} = \frac{225}{2}.$$

Collecting terms, we have

$$\frac{\gamma_1^2}{72\pi}\int_{-\infty}^{\infty} e^{-x^2} He_5^2 dx = \frac{\gamma_1^2}{72\sqrt{\pi}} \cdot \left[\frac{945}{32} - \frac{525}{4} + \frac{1950}{8} - \frac{900}{4} + \frac{225}{2}\right].$$

As before the terms in the brackets cancel out except the first one so we arrive at

$$\frac{\gamma_1^2}{72\pi}\int_{-\infty}^{\infty} e^{-x^2} He_5^2 dx = \frac{945}{72 \cdot 32\sqrt{\pi}} \cdot \gamma_1^2 = \frac{3^3 \cdot 5 \cdot 7}{2^3 \cdot 3^2 \cdot 2^5\sqrt{\pi}}\gamma_1^2 = \frac{105}{2^8\sqrt{\pi}}\gamma_1^2. \tag{3}$$

Combining results, we have obtained:

by applying a 2nd-order Gram-Charlier type A expansion, the Roughness of a distribution with standard deviation equal to 1 is approximated by

$$R[f^{(2)}] \approx R[\phi^{(2)}] + \frac{105}{2^8\sqrt{\pi}}\gamma_1^2 + \frac{35}{2^7\sqrt{\pi}}\gamma_2 + \frac{1155}{2^{13}\sqrt{\pi}}\gamma_2^2. \tag{4}$$

With accuracy up to 4th decimal, this gives

$$R[f^{(2)}] \approx R[\phi^{(2)}] + 0.2314\gamma_1^2 + 0.1543\gamma_2 + 0.0795\gamma_2^2.$$

5. Roughness for the Logistic distribution

Consider the Logistic distribution function,

$$\Lambda(x/s) = \frac{1}{1 + \exp\{-x/s\}} \equiv \Lambda.$$

The corresponding density is

$$f(x) = \frac{d\Lambda}{dx} = \frac{1}{s}\Lambda(1-\Lambda) = \frac{1}{s}(\Lambda - \Lambda^2) = \frac{1}{s}\frac{\exp\{-x/s\}}{(1 + \exp\{-x/s\})^2}.$$

Then

$$f^{(1)}(x) = \frac{1}{s} \cdot [f(x) - 2\Lambda \cdot f(x)] = \frac{1}{s} f(x)[1 - 2\Lambda].$$

$$f^{(2)}(x) = \frac{1}{s} \cdot \left[f^{(1)}(x)[1 - 2\Lambda] - 2[f(x)]^2\right] = \frac{1}{s} \cdot \left[\frac{1}{s} f(x)[1 - 2\Lambda][1 - 2\Lambda] - 2[f(x)]^2\right]$$

$$= \frac{1}{s} f(x) \cdot \left[\frac{1}{s}(1 - 4\Lambda + 4\Lambda^2) - 2\frac{1}{s}(\Lambda - \Lambda^2)\right]$$

$$\implies f^{(2)}(x) = \frac{1}{s^2} f(x) \cdot \left[1 - 6\Lambda + 6\Lambda^2\right].$$

Squaring,

$$\left[f^{(2)}(x)\right]^2 = \frac{1}{s^4}[f(x)]^2 \cdot \left[1 - 6\Lambda + 6\Lambda^2\right]^2 = \frac{1}{s^4}[f(x)]^2 \cdot [(1 - 6\Lambda)^2 + 2 \cdot (1 - 6\Lambda) \cdot 6\Lambda^2 + 36\Lambda^4]$$

$$= \frac{1}{s^4}[f(x)]^2 \cdot [1 - 12\Lambda + 36\Lambda^2 + (12 - 72\Lambda)\Lambda^2 + 36\Lambda^4]$$

$$\implies \left[f^{(2)}(x)\right]^2 = \frac{1}{s^4}[f(x)]^2 \cdot [1 - 12\Lambda + 48\Lambda^2 - 72\Lambda^3 + 36\Lambda^4].$$

Now, using the analytical expressions for the Logistic distribution and density functions, we have

$$[f(x)]^2 = \frac{1}{s^2} \frac{\exp\{-2x/s\}}{(1 + \exp\{-x/s\})^4}, \quad \Lambda^q = \frac{1}{(1 + \exp\{-x/s\})^q},$$

$$\implies [f(x)]^2 \cdot \Lambda^q = \frac{1}{s^2} \frac{\exp\{-2x/s\}}{(1 + \exp\{-x/s\})^{4+q}}.$$

Together with their constant terms, such are the integrands that we have to integrate. Specifically we have

$$\begin{aligned}\left[f^{(2)}(x)\right]^2 =& \frac{1}{s^6}\frac{\exp\{-2x/s\}}{(1+\exp\{-x/s\})^4} - \frac{12}{s^6}\frac{\exp\{-2x/s\}}{(1+\exp\{-x/s\})^5} \\ &+ \frac{48}{s^6}\frac{\exp\{-2x/s\}}{(1+\exp\{-x/s\})^6} - \frac{72}{s^6}\frac{\exp\{-2x/s\}}{(1+\exp\{-x/s\})^7} \\ &+ \frac{36}{s^6}\frac{\exp\{-2x/s\}}{(1+\exp\{-x/s\})^8}.\end{aligned}$$

In Gradshteyn & Ryzhik (2007) p. 335, eq. 3.314 we find,

$$\int_{-\infty}^{\infty}\frac{\exp\{-\mu x\}dx}{(\exp\{\beta/\gamma\}+\exp\{-x/\gamma\})^{\nu}} = \gamma\cdot\exp\{\beta(\mu-\nu/\gamma)\}\cdot \mathrm{B}(\gamma\mu,\nu-\gamma\mu), \quad \text{s.t.} \quad \frac{\nu}{\gamma} > \mu > 0.$$

Mapping coefficients to our case, we have

$$\beta = 0,\ \gamma = s, \mu = 2/s,\ \nu = q = 4,5,6,7,8$$

. The constraint is satisfied, and more over, we have $\gamma\mu = 2$. So we get

$$\int_{-\infty}^{\infty}\frac{\exp\{-2x/s\}dx}{(1+\exp\{-x/s\})^q} = s\cdot\mathrm{B}(2,q-2) = s\cdot\frac{\Gamma(2)\Gamma(q-2)}{\Gamma(2+\mathrm{q}-2)} = s\cdot\frac{\Gamma(q-2)}{\Gamma(\mathrm{q})} = s\frac{(q-3)!}{(q-1)!} = \frac{s}{(q-2)(q-1)}$$

We then get

$$R[f^{(2)}(x)] = \int_{-\infty}^{\infty}\left[f^{(2)}(x)\right]^2dx = \frac{1}{s^5}\cdot\left(\frac{1}{2\cdot 3} - 12\frac{1}{3\cdot 4} + 48\frac{1}{4\cdot 5} - 72\frac{1}{5\cdot 6} + 36\frac{1}{6\cdot 7}\right)$$

$$= \frac{1}{s^5}\cdot\left(\frac{1}{6} - 1 + \frac{12}{5} - \frac{12}{5} + \frac{6}{7}\right) = \frac{1}{s^5}\left(\frac{1}{6} - \frac{1}{7}\right)$$

$$\implies R[f^{(2)}(x)] = \frac{1}{42s^5}\,.$$

For unitary variance this becomes

$$R[f_1^{(2)}(x)] = \frac{\pi^5}{42 \cdot 3^{5/2}} \approx 0.467,$$

and the optimal bandwidth factor given the Normal Kernel is

$$h_{opt}^{N-\Lambda} = 0.776 \cdot (0.467)^{-1/5} \approx 0.903.$$

5.1. **Other measures of interest.** . Evaluated at the mode the Logistic density gives

$$f(\text{mode}) = f(0) = \frac{1}{s}\frac{\exp\{-0/s\}}{(1+\exp\{-0/s\})^2} = \frac{1}{4s}.$$

For unitary variance, we have $s = \sqrt{3}/\pi$ and so

$$f_1(\text{mode}) = \frac{\pi}{4\sqrt{3}} \approx 0.453.$$

The Quantile function of the Logistic distribution is

$$Q(p) = -s\pi \ln\left(\frac{1-p}{p}\right).$$

So the IQR for unitary variance is

$$\text{IQR} = Q(0.75) - Q(0.25) = \frac{\sqrt{3}}{\pi}\left[-\ln\left(\frac{1/4}{3/4}\right) + \ln\left(\frac{3/4}{1/4}\right)\right] = \frac{\sqrt{3}}{\pi} 2\ln 3 \approx 1.211.$$

6. m-order Roughness for the Laplace distribution

We want to compute

$$R[f^{(m)}(x)] = \int_{-\infty}^{\infty} \left[f^{(m)}(x)\right]^2 dx$$

where $f(x)$ is a density and m denotes its m-th derivative.

Because the Laplace distribution does not have a derivative at $x = 0$ we write its density (for zero-mean) in branches,

$$f(x) = \begin{cases} \frac{1}{2b} \exp\{x/b\} & x < 0 \\ \frac{1}{2b} & x = 0 \\ \frac{1}{2b} \exp\{-x/b\} & x > 0. \end{cases}$$

We can ignore the middle branch which is a single point. Nevertheless by continuity we can consider 0 as an integral limit so we have

$$R[f^{(m)}(x)] = \int_{-\infty}^{0} \left[f^{(m)}(x)\right]^2 dx \; + \; \int_{0}^{\infty} \left[f^{(m)}(x)\right]^2 dx.$$

Per integral,

$$x < 0, \qquad f^{(m)}(x) = \frac{1}{b^m} \frac{1}{2b} \exp\{x/b\} \implies \left[f^{(m)}(x)\right]^2 = \frac{1}{4b^{2m+2}} \exp\{2x/b\},$$

and

$$\int_{-\infty}^{0} \left[f^{(m)}(x)\right]^2 dx = \frac{1}{4b^{2m+2}} \int_{-\infty}^{0} \exp\{2x/b\} dx = \frac{1}{4b^{2m+2}} \int_{-\infty}^{0} \frac{b}{2} \frac{d \exp\{2x/b\}}{dx} dx$$

$$= \frac{1}{8b^{2m+1}} \cdot \left[\exp\{2x/b\} \Big|_{-\infty}^{0} \right] = \frac{1}{8b^{2m+1}}.$$

For the other integral we have

$$x > 0, \qquad f^{(m)}(x) = \frac{(-1)^m}{b^m} \frac{1}{2b} \exp\{-x/b\} \implies \left[f^{(m)}(x)\right]^2 = \frac{1}{4b^{2m+2}} \exp\{-2x/b\}$$

and

$$\int_{0}^{\infty} \left[f^{(m)}(x)\right]^2 dx = \frac{1}{4b^{2m+2}} \int_{0}^{\infty} \exp\{-2x/b\} dx = \frac{1}{4b^{2m+2}} \int_{0}^{\infty} \frac{-b}{2} \frac{d \exp\{-2x/b\}}{dx} dx$$

$$= \frac{-1}{8b^{2m+1}} \cdot \left[\exp\{-2x/b\} \Big|_0^\infty \right] = \frac{1}{8b^{2m+1}}.$$

So, overall, for order of differentiation m we have for the Laplace density,

$$R[f^{(m)}(x)] = \frac{1}{8b^{2m+1}} + \frac{1}{8b^{2m+1}} = \frac{1}{4b^{2m+1}}.$$

The scale parameter b is important here: if $b < 1$ (implying that $\text{Var}(x) < 2$), m-order Roughness will be increasing in m, while if $b > 1$ ($\text{Var}(x) > 2$) it will decrease. If $b = 1$, we have $R = 1/4$ for any order of differentiation.

For $m = 2$ we obtain $R[f^{2)}(x)] = \frac{1}{4b^5}$. For unitary variance we must set $b = 1/\sqrt{2}$ and we get

$$R[f_1^{(2)}(x)] = \frac{2^{5/2}}{4} = \sqrt{2} \approx 1.414,$$

and the optimal bandwidth factor given the Normal Kernel is

$$h_{opt}^{N-L} = 0.776 \cdot (1.414)^{-1/5} \approx 0.724.$$

6.1. **Other measures of interest.** Evaluated at the mode for unitary variance, the Laplace density gives

$$f_1\text{mode} = f_1(0) = \frac{\sqrt{2}}{2} = 1/\sqrt{2} \approx 0.707.$$

The quantile function of the zero-mean Laplace is

$$Q(p) = \begin{cases} b\ln(2p) & p \leq 1/2 \\ -b\ln(2-2p) & p \geq 1/2. \end{cases}$$

So the IQR for unitary variance is

$$\text{IQR} = -\frac{1}{\sqrt{2}}\ln(1/2) - \frac{1}{\sqrt{2}}\ln(1/2) = \frac{2\ln 2}{\sqrt{2}} \approx 0.980.$$

7. Roughness for Student's-t distribution

With p denoting the (integer) degrees of freedom, the density of Student's-t distribution (with implied scaled parameter equal to unity) can be written

$$f(x) = A_0 \cdot (p + x^2)^{-(p+1)/2}, \quad A_0 = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \frac{p^{(p+1)/2}}{\sqrt{p\pi}}.$$

We have

$$\begin{aligned}
f^{(1)}(x) &= f(x) \cdot \left(-\frac{p+1}{2}\right) \cdot (p+x^2)^{-1} \cdot 2x = -(p+1) \cdot f(x) \cdot \big[(p+x^2)^{-1} \cdot x\big]. \\
f^{(2)}(x) &= -(p+1) \cdot \Big\{ f^{(1)}(x)(p+x^2)^{-1} \cdot x + f(x) \cdot \big[-(p+x^2)^{-2} \cdot 2x \cdot x + (p+x^2)^{-1} \big] \Big\} \\
&= -(p+1) \cdot \Big\{ -(p+1) \cdot f(x) \cdot \big[(p+x^2)^{-1} \cdot x\big](p+x^2)^{-1} \cdot x \\
&\qquad + f(x) \cdot \big[-(p+x^2)^{-2} \cdot 2x \cdot x + (p+x^2)^{-1} \big] \Big\} \\
&= -(p+1) \cdot f(x) \cdot \Big\{ -(p+1) \cdot (p+x^2)^{-2} \cdot x^2 - (p+x^2)^{-2} \cdot 2x^2 + (p+x^2)^{-1} \Big\} \\
&= (p+1) \cdot f(x) \cdot (p+x^2)^{-1} \Big\{ (p+1) \cdot (p+x^2)^{-1} \cdot x^2 + (p+x^2)^{-1} \cdot 2x^2 - 1 \Big\} \\
&= (p+1) \cdot f(x) \cdot (p+x^2)^{-1} \Big[(p+3) \cdot (p+x^2)^{-1} \cdot x^2 - 1 \Big].
\end{aligned}$$

Then

$$\begin{aligned}
\big[f^{(2)}(x)\big]^2 &= (p+1)^2 \cdot f(x)^2 \cdot (p+x^2)^{-2} \Big[(p+3) \cdot (p+x^2)^{-1} \cdot x^2 - 1 \Big]^2 \\
&= (p+1)^2 \cdot f(x)^2 \cdot (p+x^2)^{-2} \Big[(p+3)^2 \cdot (p+x^2)^{-2} \cdot x^4 - 2\,(p+3) \cdot (p+x^2)^{-1} \cdot x^2 + 1 \Big] \\
&= (p+3)^2 \cdot (p+1)^2 \cdot f(x)^2 \cdot (p+x^2)^{-4} \cdot x^4 - 2\,(p+1)^2\,(p+3) \cdot f(x)^2 \cdot (p+x^2)^{-3} \cdot x^2 \\
&\qquad + (p+1)^2 \cdot f(x)^2 \cdot (p+x^2)^{-2}.
\end{aligned}$$

Now,

$$f(x)^2 = A_0^2 \cdot (p+x^2)^{-(p+1)},$$

so

$$\left[f^{(2)}(x)\right]^2 = A_1 \cdot \frac{x^4}{(p+x^2)^{p+5}} - A_2 \cdot \frac{x^2}{(p+x^2)^{p+4}} + A_3 \cdot \frac{1}{(p+x^2)^{p+3}}$$

$$A_1 = A_0^2\,(p+3)^2 \cdot (p+1)^2\,, \quad A_2 = 2A_0^2\,(p+1)^2\,(p+3)\,, \quad A_3 = A_0^2\,(p+1)^2\,.$$

We have to compute integrals of the form

$$I_{s,q} = \int_{-\infty}^{\infty} \frac{x^s}{(p+x^2)^q}dx,$$

and in our case, $s = 0, 2, 4$, so the integrand is an even function in all our cases. Therefore we have

$$I_{s,q} = 2\int_{0}^{\infty} \frac{x^s}{(p+x^2)^q}dx.$$

In Gradshteyn & Ryzhik (2007) p. 325, eq. 3.251(4.) we find

$$\int_0^\infty \frac{x^{2m}dx}{(c+ax^2)^n} = \frac{(2m-1)!!\,(2n-2m-3)!!\,\pi}{2\cdot(2n-2)!!\,a^m c^{n-m-1}\sqrt{ac}}, \quad s.t.\ a>0,\ c>0,\ n>m+1.$$

This maps to our case for $a = 1, c = p, n = q, m = s/2 \implies m = 0, 1, 2$. The constraints are all satisfied. We can also eliminate the "2" outside our integral with the "2" in the denominator in the G+R expression. Note that for $2m = s = 0$, we have $(2m-1)!! = (-1)!! = 1$ (double factorials for odd negative numbers are defined.) We get

$$I_{s,q} = \frac{(s-1)!!\,(2q-s-3)!!\,\pi}{(2q-2)!!\,p^{q-s/2-1}\sqrt{p}},$$

and so

$$I_{4,p+5} = \frac{(4-1)!!\,(2(p+5)-4-3)!!\,\pi}{(2(p+5)-2)!!\,p^{(p+5)-4/2-1}\sqrt{p}} = \frac{3!!\,(2p+3)!!\,\pi}{(2p+8)!!\,p^{p+2}\sqrt{p}} = \frac{3\pi\cdot(2p+3)!!}{(2p+8)!!\,p^{p+5/2}},$$

$$I_{2,p+4} = \frac{(2-1)!!\,(2(p+4)-2-3)!!\,\pi}{(2(p+4)-2)!!\,p^{(p+4)-2/2-1}\sqrt{p}} = \frac{\pi\cdot(2p+3)!!}{(2p+6)!!\,p^{p+5/2}},$$

$$I_{0,p+3} = \frac{(0-1)!!\,(2(p+3)-0-3)!!\,\pi}{(2(p+3)-2)!!\,p^{(p+3)-0/2-1}\sqrt{p}} = \frac{\pi\cdot(2p+3)!!}{(2p+4)!!\,p^{p+5/2}}.$$

Note the common factors. Bringing it all together, the Roughness of the Student's-t distribution is

$$R\left[f^{(2)}(x)\right] = A_1\cdot I_{4,p+5} - A_2\cdot I_{2,p+4} + A_3\cdot I_{0,p+3}$$

$$A_1 = A_0^2\,(p+3)^2\cdot(p+1)^2\,,\quad A_2 = 2A_0^2\,(p+1)^2\,(p+3)\,,\quad A_3 = A_0^2\,(p+1)^2\,.$$

Taking out all common factors we get

$$R\left[f^{(2)}(x)\right] = A_0^2\,(p+1)^2\,\frac{\pi\cdot(2p+3)!!}{p^{p+5/2}}\cdot\left[\frac{3\,(p+3)^2}{(2p+8)!!} - \frac{2\,(p+3)}{(2p+6)!!} + \frac{1}{(2p+4)!!}\right],$$

$$A_0 = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\frac{p^{(p+1)/2}}{\sqrt{p\pi}}.$$

Next we show that the last two terms inside the brackets cancel off. For the double factorial of an even number we have the relation

$$(2k)!! = 2^k\cdot k!.$$

Using this, we have

$$-\frac{2\,(p+3)}{(2p+6)!!}+\frac{1}{(2p+4)!!}=-\frac{2\,(p+3)}{(2[p+3])!!}+\frac{1}{(2[p+2])!!}=-\frac{2\,(p+3)}{2^{p+3}\cdot(p+3)!}+\frac{1}{2^{p+2}\cdot(p+2)!}=0.$$

We can simplify further since

$$A_0^2=\left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\right)^2\frac{p^{(p+1)}}{p\pi}=\left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\right)^2\frac{p^p}{\pi}.$$

We arrive at

$$R\Big[f^{(2)}(x)\Big]=\frac{(2p+3)!!}{(2p+8)!!}\cdot\left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\right)^2\cdot\frac{3\,(p+3)^2\,(p+1)^2}{p^{5/2}}.$$

The variance of Student's $t(p)$ is $\sigma^2=p/(p-2)$. So the roughness for the standardized distribution is

$$\begin{aligned}R\Big[f_1^{(2)}(x)\Big]=\sigma^5R\Big[f^{(2)}(x)\Big]&=\frac{(2p+3)!!}{(2p+8)!!}\cdot\left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\right)^2\cdot\frac{3\,(p+3)^2\,(p+1)^2}{p^{5/2}}\frac{p^{5/2}}{(p-2)^{5/2}}\\&=\frac{(2p+3)!!}{(2p+8)!!}\cdot\left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\right)^2\cdot\frac{3\,(p+3)^2\,(p+1)^2}{(p-2)^{5/2}}\end{aligned}$$

For $p=5$,

$$R\left(f_1^{(2)}\mid p=5\right)=\frac{(13)!!}{(18)!!}\cdot\left(\frac{\Gamma\,(3)}{\Gamma\,(5/2)}\right)^2\cdot\frac{3\cdot 8^2\cdot 6^2}{3^{5/2}}=0.730.$$

So the optimal bandwidth factor is

$$h_{opt}^{N-t(p)}=0.776\cdot(0.730)^{-1/5}\approx 0.826.$$

7.1. **Other measures of interest.** Evaluated at the mode for unitary variance, the Student's $t(5)$ density gives

$$f_1(\text{mode}) = \sqrt{5/3}\cdot f(0) = \sqrt{5/3}\frac{\Gamma(3)}{\Gamma(5/2)}\frac{5^3}{\sqrt{5\pi}}\cdot(5)^{-3} = \sqrt{5/3}\frac{2\cdot 4}{3\sqrt{\pi}}\frac{1}{\sqrt{5\pi}} \approx 0.490$$

The IQR for unitary variance is

$$\text{IQR} = \sqrt{3/5}\cdot[Q_5(0.75) - Q_5(0.25)] \approx 1.125,$$

as computed by software.

8. An Asymmetric Laplace distribution

Consider two independent Exponential random variables Z_1, Z_2 with scale parameters $\sigma_1 = \theta/\tau,\ \sigma_2 = \theta/(1-\tau)$ respectively, $\theta > 0$, $\tau \in (0,1)$. Their densities therefore are

$$f_{z_1}(z_1) = \frac{\tau}{\theta}\exp\left\{-\frac{\tau}{\theta}z_1\right\}, \qquad f_{z_2}(z_2) = \frac{1-\tau}{\theta}\exp\left\{-\frac{1-\tau}{\theta}z_2\right\}.$$

We want to derive the density function of their difference $X = Z_1 - Z_2$. To determine the needed convolution, we note that

$$x = z_1 - z_2 \implies z_2 = z_1 - x \geq 0 \implies z_1 \geq x.$$

This constraint is not binding when $x < 0$, but it binds when $x > 0$, affecting the limits of integration. Consequently, the convolution is to be broken in two. For

$$\begin{aligned}
x \leq 0 : f_x(x) &= \int_0^\infty f_{z_1}(z_1) f_{z_2}(z_1 - x)dz_1 = \frac{\tau(1-\tau)}{\theta^2}\int_0^\infty \exp\left\{-\frac{\tau}{\theta}z_1\right\}\exp\left\{-\frac{1-\tau}{\theta}(z_1 - x)\right\}dz_1 \\
&= \frac{\tau(1-\tau)}{\theta^2}\exp\left\{\frac{1-\tau}{\theta}x\right\}\int_0^\infty \exp\left\{-\frac{1}{\theta}z_1\right\}dz_1 \\
&= \frac{\tau(1-\tau)}{\theta^2}\exp\left\{\frac{1-\tau}{\theta}x\right\}\theta\int_0^\infty \frac{1}{\theta}\exp\left\{-\frac{1}{\theta}z_1\right\}dz_1 \\
&= \frac{\tau(1-\tau)}{\theta}\exp\left\{\frac{1-\tau}{\theta}x\right\}.
\end{aligned}$$

For

$$x > 0 : f_x(x) = \int_x^\infty f_{z_1}(z_1) f_{z_2}(z_1 - x) dz_1 = \frac{\tau(1-\tau)}{\theta^2} \int_x^\infty \exp\left\{-\frac{\tau}{\theta} z_1\right\} \exp\left\{-\frac{1-\tau}{\theta}(z_1 - x)\right\} dz_1$$

$$= \frac{\tau(1-\tau)}{\theta^2} \exp\left\{\frac{1-\tau}{\theta} x\right\} \int_x^\infty \exp\left\{-\frac{1}{\theta} z_1\right\} dz_1$$

$$= \frac{\tau(1-\tau)}{\theta^2} \exp\left\{\frac{1-\tau}{\theta} x\right\} \theta \int_x^\infty \frac{1}{\theta} \exp\left\{-\frac{1}{\theta} z_1\right\} dz_1$$

$$= \frac{\tau(1-\tau)}{\theta} \exp\left\{\frac{1-\tau}{\theta} x\right\} \exp\left\{-\frac{1}{\theta} x\right\}$$

$$= \frac{\tau(1-\tau)}{\theta} \exp\left\{\frac{-\tau}{\theta} x\right\}.$$

Combining, the density of $X = Z_1 - Z_2$ is

$$f(x) = \frac{\tau(1-\tau)}{\theta} \cdot \begin{cases} \exp\left\{\frac{1-\tau}{\theta} x\right\} & x \leq 0 \\ \exp\left\{-\frac{\tau}{\theta} x\right\} & x > 0. \end{cases}$$

For $\tau = 1/2$ we recover the Laplace distribution with scale parameter $b = 2\theta$.

The moments of the distribution are obtained by using its stochastic representation and the properties of cumulants, so

$$E(X) = \kappa_1(x) = \kappa_1(z_1) - \kappa(z_2) = \frac{\theta}{\tau} - \frac{\theta}{1-\tau} = \frac{\theta(1-\tau) - \theta\tau}{\tau(1-\tau)} = \frac{\theta(1-2\tau)}{\tau(1-\tau)},$$

$$\operatorname{Var}(X) = \sigma^2 = \kappa_2(x) = \kappa_2(z_1) + \kappa_2(z_2) = \left(\frac{\theta}{\tau}\right)^2 + \left(\frac{\theta}{1-\tau}\right)^2 = \frac{\theta^2[\tau^2 + (1-\tau)^2]}{\tau^2(1-\tau)^2},$$

$$\kappa_3(x) = \kappa_3(z_1) - \kappa_3(z_2) = 2\left(\frac{\theta}{\tau}\right)^3 - 2\left(\frac{\theta}{1-\tau}\right)^3 = 2\theta^3 \frac{(1-\tau)^3 - \tau^3}{\tau^3(1-\tau)^3}.$$

Consequently, the skewness coefficient is

$$\gamma_1(x) = \frac{\kappa_3(x)}{\kappa_2(x)^{3/2}} = 2\frac{(1-\tau)^3 - \tau^3}{([\tau^2 + (1-\tau)^2])^{3/2}}.$$

8.1. **Roughness.** The density branches being exponential, computing their 2nd derivatives is straightforward. The roughness expression is

$$\begin{aligned}
R[f^{(2)}] &= \frac{\tau^2(1-\tau)^6}{\theta^6}\int_{-\infty}^{0}\exp\left\{2\frac{1-\tau}{\theta}x\right\}dx + \frac{\tau^6(1-\tau)^2}{\theta^6}\int_{0}^{\infty}\exp\left\{-2\frac{\tau}{\theta}x\right\}dx \\
&= \frac{\tau^2(1-\tau)^6}{\theta^6}\frac{\theta}{2(1-\tau)} + \frac{\tau^6(1-\tau)^2}{\theta^6}\frac{\theta}{2\tau} \\
&= \frac{\tau^2(1-\tau)^2[(1-\tau)^3+\tau^3]}{2\theta^5}.
\end{aligned}$$

To obtain the roughness for unitary variance we have to multiply by σ^5,

$$\begin{aligned}
R[f_1^{(2)}] = \sigma^5 \cdot R[f^{(2)}] &= \frac{\tau^2(1-\tau)^2[(1-\tau)^3+\tau^3]}{2\theta^5} \cdot \left(\frac{\theta^2(1-2\tau+2\tau^2)}{\tau^2(1-\tau)^2}\right)^{5/2} \\
&= \frac{(1-\tau)^3+\tau^3}{2\tau^3(1-\tau)^3} \cdot \left[\tau^2+(1-\tau)^2\right]^{5/2}
\end{aligned}$$

8.2. **Other measures of interest.** The distribution has its mode at zero. Its value for unitary variance is

$$f_1(\text{mode})) = \sigma f(0) = \frac{\theta\sqrt{\tau^2+(1-\tau)^2}}{\tau(1-\tau)}\frac{\tau(1-\tau)}{\theta} = \sqrt{\tau^2+(1-\tau)^2}$$

.

Interquantile Range. The distribution function has also branches.

$$F(x) = \begin{cases} \frac{\tau(1-\tau)}{\theta}\int_{-\infty}^{x}\exp\left\{\frac{1-\tau}{\theta}t\right\}dt & x \le 0 \\ \\ F(0) + \frac{\tau(1-\tau)}{\theta}\int_{0}^{x}\exp\left\{-\frac{\tau}{\theta}t\right\} & x > 0. \end{cases}$$

$$\implies F(x) = \begin{cases} \tau\exp\left\{\frac{1-\tau}{\theta}x\right\} & x \le 0 \\ \\ \tau + (1-\tau)\left[1-\exp\left\{-\frac{\tau}{\theta}x\right\}\right] & x > 0. \end{cases}$$

$$\Longrightarrow F(x) = \begin{cases} \tau \exp\left\{\frac{1-\tau}{\theta} x\right\} & x \leq 0 \\ \\ 1 - (1-\tau)\exp\left\{-\frac{\tau}{\theta} x\right\}] & x > 0. \end{cases}$$

Consequently the Quantile function is

$$Q(p) = \begin{cases} \frac{\theta}{1-\tau} \ln(p/\tau) & p \leq \tau \\ \\ \frac{\theta}{\tau} \ln\left(\frac{1-\tau}{1-p}\right) & p > \tau. \end{cases}$$

For unitary variance we divide by the standard deviation,

$$Q(p \mid \sigma = 1) = \frac{\tau(1-\tau)}{\theta\sqrt{\tau^2 + (1-\tau)^2}} \begin{cases} \frac{\theta}{1-\tau} \ln(p/\tau) & p \leq \tau \\ \\ \frac{\theta}{\tau} \ln\left(\frac{1-\tau}{1-p}\right) & p > \tau. \end{cases}$$

$$Q(p \mid \sigma = 1) = \begin{cases} \frac{\tau \ln(p/\tau)}{\sqrt{\tau^2+(1-\tau)^2}} & p \leq \tau \\ \\ \frac{(1-\tau)}{\sqrt{\tau^2+(1-\tau)^2}} \ln\left(\frac{1-\tau}{1-p}\right) & p > \tau. \end{cases}$$

With this we can compute the IQR, depending on the value of τ.

9. AMISE with misspecification and sample standard deviation

The non-optimized theoretical AMISE expression is (eq. (4) in the main text)

$$\text{AMISE}\left\{\widehat{f}(x)\right\} \approx \frac{\kappa_2^2(k)}{4} h^4 R\left(f^{(2)}\right) + \frac{R(k)}{nh}.$$

By taking the derivative with respect to h and setting it equal to zero, we obtain the optimal bandwidth

$$h_{opt}^{k-f} = \left[\frac{R\left(k\right)}{\kappa_2^2(k)}\right]^{1/5} \left[R\left(f^{(2)}\right)\right]^{-1/5} n^{-1/5}.$$

But suppose now that we have misspecified the kernel density f, and used instead some g density. In this cases we would obtain the "optimal" bandwidth as

$$h_{opt}^{k-g} = \left[\frac{R\left(k\right)}{\kappa_2^2(k)}\right]^{1/5} \left[R\left(g^{(2)}\right)\right]^{-1/5} n^{-1/5}.$$

Inserting this into the expression for the AMISE, we get

$$\begin{aligned}\text{AMISE}_{k-g}^{k-f} &\approx \frac{\kappa_2^2(k)}{4}\left[\frac{R\left(k\right)}{\kappa_2^2(k)}\right]^{4/5} \left[R\left(g^{(2)}\right)\right]^{-4/5} n^{-4/5} R\left(f^{(2)}\right) \\ &+ \frac{R\left(k\right)}{n}\left[\frac{R\left(k\right)}{\kappa_2^2(k)}\right]^{-1/5} \left[R\left(g^{(2)}\right)\right]^{1/5} n^{1/5} \\ &= R(k)^{4/5}\kappa_2^{2/5}\left(\frac{R\left(f^{(2)}\right)}{4R\left(g^{(2)}\right)^{4/5}} + R\left(g^{(2)}\right)^{1/5}\right) n^{-4/5}.\end{aligned}$$

Assume first that we know the true standard deviation of the population, σ. Then using $R\left(f^{(2)}\right) = \sigma^{-5} R\left(f_1^{(2)}\right)$ and likewise for the g density, we get

$$\begin{aligned}\text{AMISE}_{k-g}^{k-f} &= R(k)^{4/5}\kappa_2^{2/5}\left(\frac{R\left(f_1^{(2)}\right)\sigma^{-5}}{4R\left(g_1^{(2)}\right)^{4/5}\sigma^{-4}} + R\left(g_1^{(2)}\right)^{1/5}\sigma^{-1}\right) n^{-4/5} \\ &= R(k)^{4/5}\kappa_2^{2/5}\left(\frac{R\left(f_1^{(2)}\right)}{4R\left(g_1^{(2)}\right)^{4/5}} + R\left(g_1^{(2)}\right)^{1/5}\right) \sigma^{-1} n^{-4/5}.\end{aligned}$$

This is equation (16) of the main text.

Suppose now that we do not know σ but we estimate it from the sample, obtaining $\widehat{\sigma}$. In the AMISE expression, this will affect only the magnitudes that depend on our actions, i.e. in relation only to the g density so we get

$$\begin{aligned}
\mathrm{AMISE}_{k-g}^{k-f} &= R(k)^{4/5}\kappa_2^{2/5}\left(\frac{R\left(f_1^{(2)}\right)\sigma^{-5}}{4R\left(g_1^{(2)}\right)^{4/5}\widehat{\sigma}^{-4}} + R\left(g_1^{(2)}\right)^{1/5}\widehat{\sigma}^{-1}\right)n^{-4/5} \\
&= R(k)^{4/5}\kappa_2^{2/5}\left(\frac{R\left(f_1^{(2)}\right)}{4R\left(g_1^{(2)}\right)^{4/5}}\left(\frac{\widehat{\sigma}}{\sigma}\right)^4 + R\left(g_1^{(2)}\right)^{1/5}\left(\frac{\sigma}{\widehat{\sigma}}\right)\right)\sigma^{-1}n^{-4/5}.
\end{aligned}$$

This is equation (18) of the main text.

References

Gradshteyn, I. S. & Ryzhik, I. M. (2007), *Tables of integrals, series and functions (7th ed)*, Academic Press.