Matrix Algebra Derivations

$$\begin{aligned} y &= \alpha + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + u \\ &= \begin{pmatrix} 1 & x_1 & x_2 & \ldots & x_k \end{pmatrix} \begin{pmatrix} \alpha \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix} + u \\ &\equiv X\delta + u \end{aligned}$$

Estimation

$$\begin{aligned} X'\widehat{u} &= 0 \\ X'\left(y - X\widehat{\delta}\right) &= 0 \\ X'y - X'X\widehat{\delta} &= 0 \\ \widehat{\delta} &= \left(X'X\right)^{-1} X'y \end{aligned}$$

Unbiasedness

$$\begin{aligned} E\left(\widehat{\delta}\right) &= E\left(\left(X'X\right)^{-1} X'y\right) \\ &= E\left(\left(X'X\right)^{-1} X'\left(X\delta + u\right)\right) \\ &= E\left(\left(X'X\right)^{-1} X'X\delta + \left(X'X\right)^{-1} X'u\right) \\ &= E\left(\delta + \left(X'X\right)^{-1} X'u\right) \\ &= \delta + E\left(\left(X'X\right)^{-1} X'u\right) \\ &= \delta \end{aligned}$$

Variance

$$\begin{aligned} V\left(\widehat{\delta}\right) &= V\left(\left(X'X\right)^{-1} X'y\right) \\ &= V\left(\left(X'X\right)^{-1} X'\left(X\delta + u\right)\right) \\ &= V\left(\left(X'X\right)^{-1} X'X\delta + \left(X'X\right)^{-1} X'u\right) \\ &= V\left(\delta + \left(X'X\right)^{-1} X'u\right) \\ &= V\left(\left(X'X\right)^{-1} X'u\right) \\ &= \sigma^2 \left(X'X\right)^{-1} \end{aligned}$$