Constrained optimization

$$\mathcal{L} = u(x,y) + \lambda\,(I - P_x x - P_y y)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial u(x,y)}{\partial x} - \lambda P_x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial u(x,y)}{\partial y} - \lambda P_y = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = (I - P_x x - P_y y) = 0$$

$$\frac{MU_x}{MU_y} = \frac{\frac{\partial u(x,y)}{\partial x}}{\frac{\partial u(x,y)}{\partial y}} = \frac{P_x}{P_y}$$

Ex: $u(x,y) = x^{\alpha} y^{\beta}$

$$\mathcal{L} = x^{\alpha} y^{\beta} + \lambda (I - P_x x - P_y y)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \alpha x^{\alpha-1} y^{\beta} - \lambda P_x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = \beta x^{\alpha} y^{\beta-1} - \lambda P_y = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = (I - P_x x - P_y y) = 0$$

$$\frac{\alpha y}{\beta x} = \frac{P_x}{P_y}$$

if $\alpha + \beta = 1 \Rightarrow \beta = 1 - \alpha$

$$P_y y = \frac{1-\alpha}{\alpha} P_x x$$

$$I = P_x x + P_y y = P_x x + \left(\frac{1-\alpha}{\alpha} P_x x\right) = \frac{1}{\alpha} P_x x$$

$$\Rightarrow x^* = \alpha \frac{I}{P_x} \quad \& \quad y^* = (1-\alpha) \frac{I}{P_y}$$