SUPPLEMENTAL APPENDIX FOR: ESTIMATION OF A VARYING COEFFICIENT, FIXED-EFFECTS COBB-DOUGLAS PRODUCTION FUNCTION IN LEVELS

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This version: January 4, 2022

Abstract: This supplemental appendix contains three sections with supporting materials for "Estimation of a varying coefficient, fixed-effects Cobb-Douglas production function in levels." Section 1 gives simulation results for the comparison between our proposed estimator and the estimator by Guo et al. (2016) with cross-sectional data. Section 2 investigates the incidental parameter problem in our model by evaluating the finite-sample performance of our estimator under large n (individual units) and small T (time units). Section 3 introduces an R package that implements the developed estimator, and provides an example that illustrates how the package can be used in practice.

Keywords: panel data; profile nonlinear least-squares; semiparametric regression; smooth coefficient

JEL Classifications: C14, C23.

1 A Comparison Study

This section compares the finite-sample performance of a special case of our proposed estimator for a semiparametric (varying coefficient) Cobb-Douglas fixed-effects (S-CD-FE) model to the estimator in Guo et al. (2016) designed for cross-sectional data (i.e., semiparametric Cobb-Douglas quasi-likelihood: S-CD-QL).[1] We consider a bivariate production model with two production inputs ($p = 2$). We generate data according to

$$Y_i = X_{i,1}^{\beta_1(Z_{i,1}^\top \gamma_1)} X_{i,2}^{\beta_2(Z_{i,2}^\top \gamma_2)} + u_i, \tag{1}$$

where $Z_j = (Z_{j1}, Z_{j2})$ are drawn independently from $U(1,4)$ for $j = 1,2$, $X_j = 0.5(Z_{j1} + Z_{j2}) + \zeta$ mimics possible correlations between inputs and environment variables with $\zeta \sim \mathcal{N}(0, 0.5)$, and we rescale X_j into the range of $[1,4]$ to ensure positive values. $u \sim \mathcal{N}(0,1)$ is drawn independently from all X and Z. We consider three DGPs with different specifications on $\beta_j(\cdot)$ and $\gamma = (\gamma_1^\top, \gamma_2^\top)^\top$, with $\gamma_j = (\gamma_{j1}, \gamma_{j2})^\top$. In DGP_1, $\beta_1(v) = \sin(v\pi/2), \beta_2(v) = -\sin(3v\pi/4)$ and $(\gamma_1, \gamma_2)^\top = (1/\sqrt{3}, \sqrt{2/3}, 2/\sqrt{5}, \sqrt{1/5})$; in DGP_2, $\beta_1(v) = \sqrt{v\pi}, \beta_2(v) = \cos(v\pi/2)$ and $(\gamma_1, \gamma_2)^\top = (1/\sqrt{3}, -\sqrt{2/3}, \sqrt{1/5}, -2/\sqrt{5})$; and in DGP_3, we have $\beta_1(v) = \exp(v/2), \beta_2(v) = v$ and $(\gamma_1, \gamma_2)^\top = (1/\sqrt{6}, \sqrt{5/6}, \sqrt{2/5}, \sqrt{3/5})$.

In our first step, we implement cubic B-spline basis functions (of order $m = 3$) and place a total of $L = J + m + 1$ knots on the support of $v_{i,j}(\gamma_j)$, where J refers to the number of (strictly) interior knots of $v_{i,j}(\gamma_j)$. We select $J = [n^{1/5}]$ as a *rule-of-thumb* (ROT) approach following the literature, where $[v]$ is the integer part of a real number v. We place the l^{th} interior knot on the $(l/J+1)^{th}$ percentile of $v_{i,j}(\gamma_j)$ for $l = 1, ..., J$. In the second step, we use a second-order Gaussian kernel function with a ROT bandwidth $h_j = \widehat{\sigma}_j n^{-1/5}$, where $\widehat{\sigma}_j$ is the empirical standard deviation of $\{v_{i,j}(\widehat{\gamma}_j)\}_{i=1}^n$. We set $n = (100, 200, 400)$ and conduct 500 repetitions. We evaluate the performance of the estimator $\widehat{\gamma}$ by our profile nonlinear least-squares (PNLS) and quasi-likelihood (QL) via root mean squared error (RMSE), absolute bias (ABIAS), and standard deviation (ASTD). We evaluate the performance of the estimator $\check{\beta}(\cdot)$ by our local-nonlinear kernel (LNK) and QL via

[1] R code for this estimator is available upon request.

Table A1: Simulation Comparison Results for S-CD-FE and S-CD-PQL

			S-CD-FE			S-CD-PQL				S-CD-FE			S-CD-PQL		
	n		RMSE	BIAS	STD	RMSE	BIAS	STD		RAMSE	ABIAS	ASTD	RAMSE	ABIAS	ASTD
DGP_1	100	$\widehat{\gamma}_{11}$	0.0462	0.0363	0.0288	0.0417	0.0068	0.0413	$\check{\beta}_1$	0.1350	0.1068	0.1001	0.1613	0.1184	0.1489
		$\widehat{\gamma}_{12}$	0.0349	0.0258	0.0236	0.0309	0.0080	0.0309	$\check{\beta}_2$	0.1547	0.1173	0.0786	0.1584	0.1014	0.1202
		$\widehat{\gamma}_{21}$	0.0392	0.0301	0.0252	0.0488	0.0378	0.0311							
		$\widehat{\gamma}_{22}$	0.0295	0.0214	0.0204	0.0373	0.0271	0.0258							
	200	$\widehat{\gamma}_{11}$	0.0328	0.0264	0.0195	0.0278	0.0046	0.0275	$\check{\beta}_1$	0.1055	0.0847	0.0731	0.1358	0.1008	0.1114
		$\widehat{\gamma}_{12}$	0.0241	0.0185	0.0155	0.0210	0.0032	0.0210	$\check{\beta}_2$	0.1279	0.0960	0.0674	0.1468	0.0857	0.0955
		$\widehat{\gamma}_{21}$	0.0243	0.0223	0.0096	0.0395	0.0298	0.0260							
		$\widehat{\gamma}_{22}$	0.0165	0.0153	0.0062	0.0305	0.0214	0.0217							
	400	$\widehat{\gamma}_{11}$	0.0222	0.0210	0.0072	0.0207	0.0031	0.0171	$\check{\beta}_1$	0.0791	0.0637	0.0508	0.1208	0.0891	0.0853
		$\widehat{\gamma}_{12}$	0.0152	0.0144	0.0047	0.0141	0.0025	0.0117	$\check{\beta}_2$	0.1041	0.0780	0.0552	0.1277	0.0731	0.0722
		$\widehat{\gamma}_{21}$	0.0200	0.0195	0.0041	0.0247	0.0227	0.0100							
		$\widehat{\gamma}_{22}$	0.0137	0.0135	0.0027	0.0168	0.0155	0.0065							
DGP_2	100	$\widehat{\gamma}_{11}$	0.1152	0.0890	0.0735	0.0748	0.0298	0.0690	$\check{\beta}_1$	0.1166	0.0855	0.1064	0.1398	0.1042	0.1287
		$\widehat{\gamma}_{12}$	0.0696	0.0573	0.0399	0.0926	0.0412	0.0958	$\check{\beta}_2$	0.1286	0.0931	0.1083	0.1519	0.0892	0.1135
		$\widehat{\gamma}_{21}$	0.0823	0.0670	0.0481	0.0992	0.0744	0.0659							
		$\widehat{\gamma}_{22}$	0.0594	0.0474	0.0361	0.0628	0.0493	0.0391							
	200	$\widehat{\gamma}_{11}$	0.0596	0.0474	0.0364	0.0681	0.0264	0.0531	$\check{\beta}_1$	0.0790	0.0571	0.0728	0.1049	0.0744	0.0934
		$\widehat{\gamma}_{12}$	0.0411	0.0333	0.0243	0.0738	0.0380	0.0849	$\check{\beta}_2$	0.1036	0.0768	0.0824	0.1164	0.0702	0.0861
		$\widehat{\gamma}_{21}$	0.0520	0.0405	0.0329	0.0624	0.0494	0.0383							
		$\widehat{\gamma}_{22}$	0.0357	0.0281	0.0222	0.0441	0.0352	0.0267							
	400	$\widehat{\gamma}_{11}$	0.0450	0.0349	0.0286	0.0327	0.0115	0.0308	$\check{\beta}_1$	0.0593	0.0435	0.0547	0.0875	0.0610	0.0696
		$\widehat{\gamma}_{12}$	0.0318	0.0244	0.0205	0.0586	0.0179	0.0561	$\check{\beta}_2$	0.0920	0.0676	0.0646	0.0967	0.0556	0.0677
		$\widehat{\gamma}_{21}$	0.0371	0.0309	0.0205	0.0436	0.0340	0.0274							
		$\widehat{\gamma}_{22}$	0.0258	0.0217	0.0142	0.0308	0.0240	0.0194							
DGP_3	100	$\widehat{\gamma}_{11}$	0.0098	0.0081	0.0056	0.0771	0.0109	0.0765	$\check{\beta}_1$	0.1398	0.1085	0.1021	0.2078	0.1452	0.1776
		$\widehat{\gamma}_{12}$	0.0113	0.0094	0.0064	0.0799	0.0052	0.0803	$\check{\beta}_2$	0.1336	0.1653	0.1186	0.1169	0.0705	0.0786
		$\widehat{\gamma}_{21}$	0.0080	0.0065	0.0046	0.0115	0.0090	0.0071							
		$\widehat{\gamma}_{22}$	0.0092	0.0075	0.0054	0.0132	0.0104	0.0081							
	200	$\widehat{\gamma}_{11}$	0.0065	0.0051	0.0040	0.0617	0.0095	0.0667	$\check{\beta}_1$	0.1044	0.0831	0.0752	0.1544	0.1003	0.1128
		$\widehat{\gamma}_{12}$	0.0075	0.0059	0.0046	0.0682	0.0034	0.0785	$\check{\beta}_2$	0.1107	0.1333	0.0956	0.1088	0.0608	0.0525
		$\widehat{\gamma}_{21}$	0.0057	0.0046	0.0033	0.0065	0.0052	0.0038							
		$\widehat{\gamma}_{22}$	0.0065	0.0053	0.0038	0.0075	0.0061	0.0045							
	400	$\widehat{\gamma}_{11}$	0.0049	0.0038	0.0031	0.0550	0.0015	0.0613	$\check{\beta}_1$	0.0753	0.0614	0.0507	0.1220	0.0869	0.0854
		$\widehat{\gamma}_{12}$	0.0056	0.0044	0.0035	0.0598	0.0016	0.0699	$\check{\beta}_2$	0.0981	0.1051	0.0778	0.0850	0.0481	0.0416
		$\widehat{\gamma}_{21}$	0.0037	0.0030	0.0022	0.0048	0.0038	0.0030							
		$\widehat{\gamma}_{22}$	0.0043	0.0035	0.0026	0.0056	0.0044	0.0034							

root averaged MSE (RAMSE), averaged BIAS (ABIAS), and averaged STD (ASTD).

Table A1 reports the simulation results. Compared to estimator in Guo et al. (2016), our S-CD-FE estimates $\widehat{\gamma}_1 = (\widehat{\gamma}_{11}, \widehat{\gamma}_{12})$ in DGP_{1-2} disclose a lower averaged variance (except for $\widehat{\gamma}_{11}$ in DGP_2 with $n = 100$), but a higher averaged bias and RMSE. In contrast, our estimates $\widehat{\gamma}_2 = (\widehat{\gamma}_{21}, \widehat{\gamma}_{22})$ outperform their S-CD-QL counterparts uniformly over all DGPs and samples. The S-CD-FE in general estimates nonparametric functions, especially β_1, with a relatively lower RAMSE and ASTD than S-CD-QL. In addition, our proposed estimator is computationally more attractive than S-CD-QL, since the latter involves a simulated finite integral for quasi-likelihood given the i^{th} pair of observations $\{Y_i, X_i^\top, Z_i^\top\}$. In our experiment, using R with Intel Xeon E3-1505M V5 4-core CPU and 64GB RAM for 500 repetitions, when $n = 100$, it takes 2.44s on average for S-CD-FE, compared to 5.41 of S-CD-QL. When $n = 400$, it takes 8.26s for S-CD-FE but 19.33s for S-CD-QL. The result suggests that our proposed estimator may serve as a viable and practical alternative to model a conditional mean function when the coefficient functions of variables are nonparametrically specified with different indexed variables. Overall, our proposed estimator exhibits promising numerical performance in both the cross-sectional and panel data settings.

2 Incidental Parameter Problem

In our model S-CD-FE

$$Y_{it} = \beta_0(Z_{it,0}^\top \gamma_0) \left(\prod_{s=1}^{p} X_{it,s}^{\beta_s(Z_{it,s}^\top \gamma_s)} \right) + u_{it}, \quad i = 1, ..., n, \ t = 1, ..., T, \tag{2}$$

estimation of the individual fixed-effects α_i, which appear in $\beta_0(\cdot)$, present a challenge when n becomes large while T remains fixed (i.e., the incidental parameters problem). In this case, we expect a worse numerical performance of our estimator because α_i is not consistently estimated.

Here we consider a sample size of large n and small T $(n, T) = (240, 5)$. As in Section 3 of the paper, we consider a bivariate production model with two production inputs by generating data

Table A2: Simulation Results for S-CD-FE with a large ratio of n/T

	(n,T)		RMSE	BIAS	STD		RAMSE	ABIAS	ASTD
DGP_1	$(240,5)$	$\widehat{\gamma}_0$: Ave	39.4951	28.8323	27.1547	$\check{\beta}_0$	12.4700	6.1354	11.0208
		$\widehat{\gamma}_{11}$	0.1044	0.0776	0.0702	$\check{\beta}_1$	1.3571	0.6240	1.1178
		$\widehat{\gamma}_{12}$	0.0914	0.0623	0.0572	$\check{\beta}_2$	2.1126	0.9189	1.8732
		$\widehat{\gamma}_{21}$	0.1552	0.0846	0.0707				
		$\widehat{\gamma}_{22}$	0.1684	0.1118	0.1066				
DGP_2	$(240,5)$	$\widehat{\gamma}_0$: Ave	42.3545	35.5256	29.1432	$\check{\beta}_0$	19.8281	11.0909	14.7096
		$\widehat{\gamma}_{11}$	0.1353	0.1061	0.0845	$\check{\beta}_1$	1.9504	0.7386	1.7874
		$\widehat{\gamma}_{12}$	0.1342	0.0935	0.0868	$\check{\beta}_2$	2.3534	1.1444	1.9303
		$\widehat{\gamma}_{21}$	0.1738	0.0987	0.0861				
		$\widehat{\gamma}_{22}$	0.1890	0.1228	0.1126				
DGP_3	$(240,5)$	$\widehat{\gamma}_0$: Ave	11.1600	6.6450	5.2561	$\check{\beta}_0$	10.0044	4.5922	8.9484
		$\widehat{\gamma}_{11}$	0.1297	0.1111	0.0972	$\check{\beta}_1$	1.2865	0.5619	1.0789
		$\widehat{\gamma}_{12}$	0.1186	0.0973	0.0781	$\check{\beta}_2$	2.0742	0.8956	1.6914
		$\widehat{\gamma}_{21}$	0.1639	0.1316	0.1182				
		$\widehat{\gamma}_{22}$	0.2120	0.1830	0.1676				

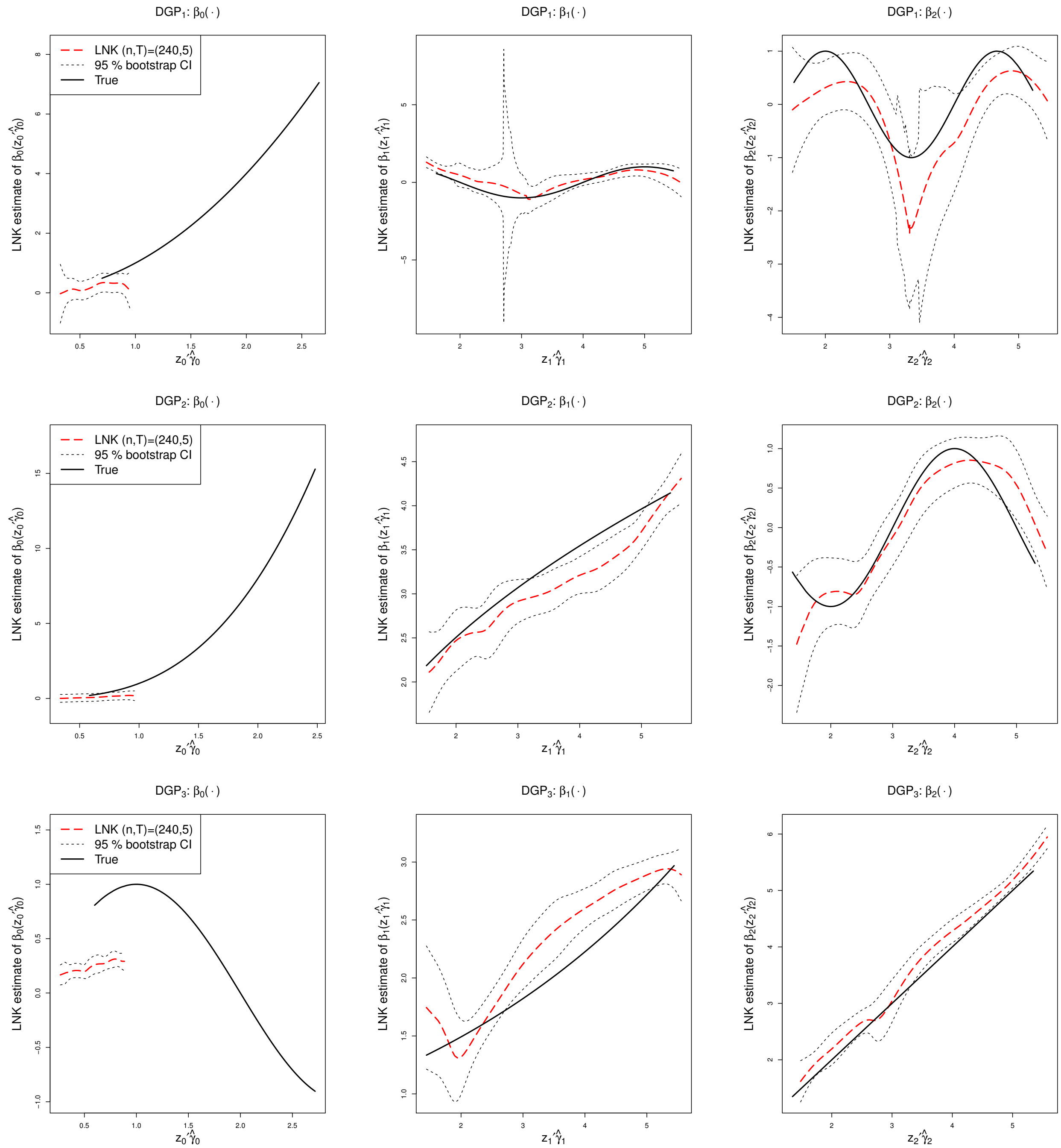
based on (2) as

$$Y_{it} = \beta_0(Z_{it,0}^\top\gamma_0)\left(X_{it,1}^{\beta_1(Z_{it,1}^\top\gamma_1)}X_{it,2}^{\beta_2(Z_{it,2}^\top\gamma_2)}\right) + u_{it}, \tag{3}$$

where $Z_j = (Z_{j1}, Z_{j2})$ are drawn independently from $U(1,4)$ for $j = 0,1,2$, $X_j = 0.5(Z_{j1}+Z_{j2})+\zeta$ mimics possible correlations between inputs and environment variables with $\zeta \sim \mathcal{N}(0, 0.5)$, and we rescale X_j into the range of $[1,4]$ to ensure positive values. The fixed-effects are generated as $\alpha_i = \frac{1}{T}\sum_{t=1}^{T} c_0(X_{it,1}+X_{it,2}+\sum_{j=0}^{2}(Z_{it,j1}+Z_{it,j2}))+\xi_i$, where $\xi_i \sim \mathcal{N}(0, 1.25^2)$ and $c_0 = 1$ implies that the fixed-effects are correlated with the regressors. $u \sim \mathcal{N}(0,1)$ is drawn independently from all X and Z. We adopt the same DGP_{1-3} from the paper with different specifications on $\beta_j(\cdot)$ and $\gamma = (\gamma_0^\top, \gamma_1^\top, \gamma_2^\top)^\top$, where $\gamma_j = (\gamma_{j1}, \gamma_{j2})^\top$. Again, we rescale γ_0 empirically to satisfy $||\gamma_0|| = 1$ in each DGP. We choose the tuning parameters J and h in the series and kernel estimation step, respectively, in the same fashion as in Section 3 of the paper. Everything else remains unchanged from Section 1 above.

Table A2 reports the simulation results. As expected, we observe a clear negative impact of incidental parameter problem on the estimator's performance. Namely, the large $n = 250$ with a small $T = 5$ leads to a much worse estimation of the fixed-effects, reflected by the inflated average of RMSE, BIAS, and STD for the index estimator of γ_0 (i.e., $\widehat{\gamma}_0$: Ave), with the largest magnitude

Figure A1: Kernel Estimation (LNK) of $\beta_j(\cdot)$ across DGP_{1-3}



in DGP_2, followed by DGP_1 and DGP_3. Compared to that in Table 1 of the paper under DGP_1 with $(n, T) = (40, 30)$, for instance, the mean of RMSE for $\widehat{\gamma}_0$ increases by almost 100 fold. The parameter estimates in coefficient functions are also significantly worsen. As a result, the estimator for coefficient functions are deteriorated the most in $\check{\beta}_0(\cdot)$ relative to $\check{\beta}_1(\cdot)$ and $\check{\beta}_2(\cdot)$ across all DGPs. This can be seen more vividly in Figure A1, where each column plots the corresponding function estimate under $(n, T) = (240, 5)$ (blue dot-dash line) against the true function (black solid line) in DGP_1 (first row), DGP_2 (second row), and DGP_3 (third row). We observe that the fixed-effects are largely underestimated in each DGP. As a consequence, the function estimate $\tilde{\beta}_0(\cdot)$ (the first column) is clearly shorten and downward scaled, highly deviating from the true function. The bias emanating from the fixed-effects estimation also impacts the coefficient function estimator, particularly those with a high degree of curvature. This may be expected because each coefficient function is backfitted by the pilot series approximation of $\beta_0(\cdot)$ in the second step, which can be inconsistent due to the fixed-effects estimation.

Overall, our results should be taken as numerical evidence for applied researchers to avoid having a large n with small T for a panel data model with fixed-effects (i.e., a large ratio of n/T) in order to achieve reasonable estimation for our model.

3 Demonstration of our R Package

The proposed estimation procedure for S-CD-FE in (2) is implemented in an **R** package with user-friendly design. The package is available at https://694160821.wixsite.com/taining (navigate to the top panel of **R code**). The code is stored in a text file and needs to be copy-pasted into an R script for it to run. The file contains six packages (on the top of the file) that need to be pre-installed before running the code for S-CD-FE estimation. Below these packages, there are a total of 10 functions defined: Function 8 generates simulated data for demonstration purpose; function 9, which uses functions 1-7, performs estimation for S-CD-FE; and function 10 summarizes the estimation result by a table and several figures.

As an example, we use function 8 to generate data according to a simple DGP as

$$Y_{it} = \beta_0(Z_{it,0}^\top \gamma_0) \left(X_{it,1}^{\beta_1(Z_{it,1}^\top \gamma_1)} X_{it,2}^{\beta_2(Z_{it,2}^\top \gamma_2)} \right) + u_{it}, \tag{4}$$

where $Z_j = (Z_{j1}, Z_{j2}) \sim U(1, 4)$ for $j = 0, 1, 2$, and $X_s \sim U(1, 4)$ is drawn independently for $s = 1, 2$

(i.e., $q_j = p = 2$). The parameters in the coefficient functions are $\gamma = (\gamma_0^\top, \gamma_1^\top, \gamma_2^\top)^\top$, with $\gamma_j = (\gamma_{j1}, \gamma_{j2})^\top$. We generate the fixed effects as $\alpha_i = \frac{1}{T}\sum_{t=1}^{T}(X_{it,1} + X_{it,2} + \sum_{j=0}^{2}(Z_{it,j1} + Z_{it,j2})) + \xi_i$, where $\xi_i \sim \mathcal{N}(0,1)$ Finally, $u \sim \mathcal{N}(0,1)$ is drawn independently from all X and Z. Recall that $\alpha_{-1} = [\alpha_2, ..., \alpha_n]^\top$ is a $(n-1)\times 1$ vector with α_1 dropped. We specify $\beta_0(v) = v^2$, $\beta_1(v) = \sin(v\pi/2), \beta_2(v) = -\sin(3v\pi/4)$, $\gamma_0 = (2, 4, \alpha_{-1}^\top)^\top$, and $(\gamma_1, \gamma_2)^\top = (1/\sqrt{3}, \sqrt{2/3}, 2/\sqrt{5}, \sqrt{1/5})$. Vector γ_0 is empirically rescaled to satisfy $||\gamma_0|| = 1$. To reduce computational cost in this demonstration, we set $(n,T) = (15, 20)$.

Let $Z = [Z_0^\top, Z_1^\top, Z_2^\top]^\top$ and $X = [X_1^\top, X_2^\top]^\top$. Implementing the following code generates the data $\{Y_{it}, X_{it}^\top, Z_{it}^\top\}_{i=1,t=1}^{n,T}$:

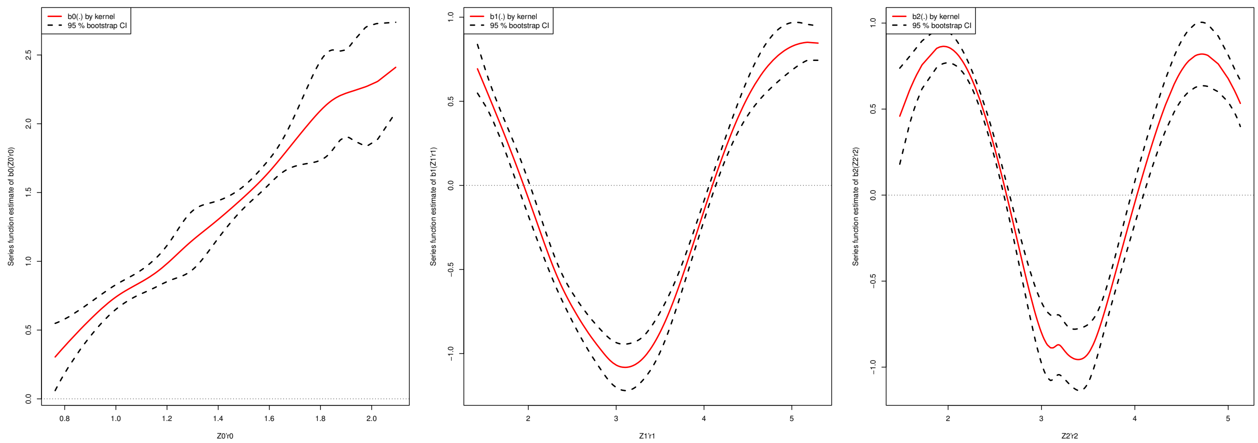
```
> set.seed(9)
> DGP<-psudo.data(n=15,t=20,a=1,b=4,c=1,d=4)
```

where n and t represent the number of individual and time units, respectively; (a,b) are the lower and upper bounds of a uniform distribution (i.e., $U(a,b)$) for all X; and (c,d) are the lower and upper bounds of a uniform distribution (i.e., $U(c,d)$) for all Z. The DGP returns a $nT \times 1$ vector y, a $nT \times 2$ matrix x, and a z.list in a list form with three dimensions, each of which contains a $nT \times 2$ matrix $\{Z_{j1,it}, Z_{j2,it}\}_{i=1,t=1}^{n,T}$ corresponding to $\beta_j(\cdot)$. Running the following code performs estimation for the model in (4):

```
> SCD.model<-WH.SCD(y=DGP$y,x=DGP$x,z.list = DGP$z.list,boot.num = 29,Kernel=TRUE)
> result.scd<-result.WH.SCD(SCD.model,t=t,figure=TRUE)
```

where WH.SCD() estimates the S-CD-FE and result.WH.SCD() reports a summary of empirical results. Namely, WH.SCD() is implemented using data from the DGP above, with 29 bootstrap repetitions for (point-wise) standard errors of function estimates (boot.num = 29). We purposely set a small boot.num in this example in order to increase computational speed, and it should be a large integer, such as 999, in practice. Notice that WH.SCD() is designed to report both series and kernel function estimates if Kernel=TRUE, and it only reports series estimates if Kernel=FALSE. Therefore, given a large sample data in practice, one can take advantage of the series estimator

Figure A2: Kernel Estimation of $\beta_j(\cdot)$ in the Package

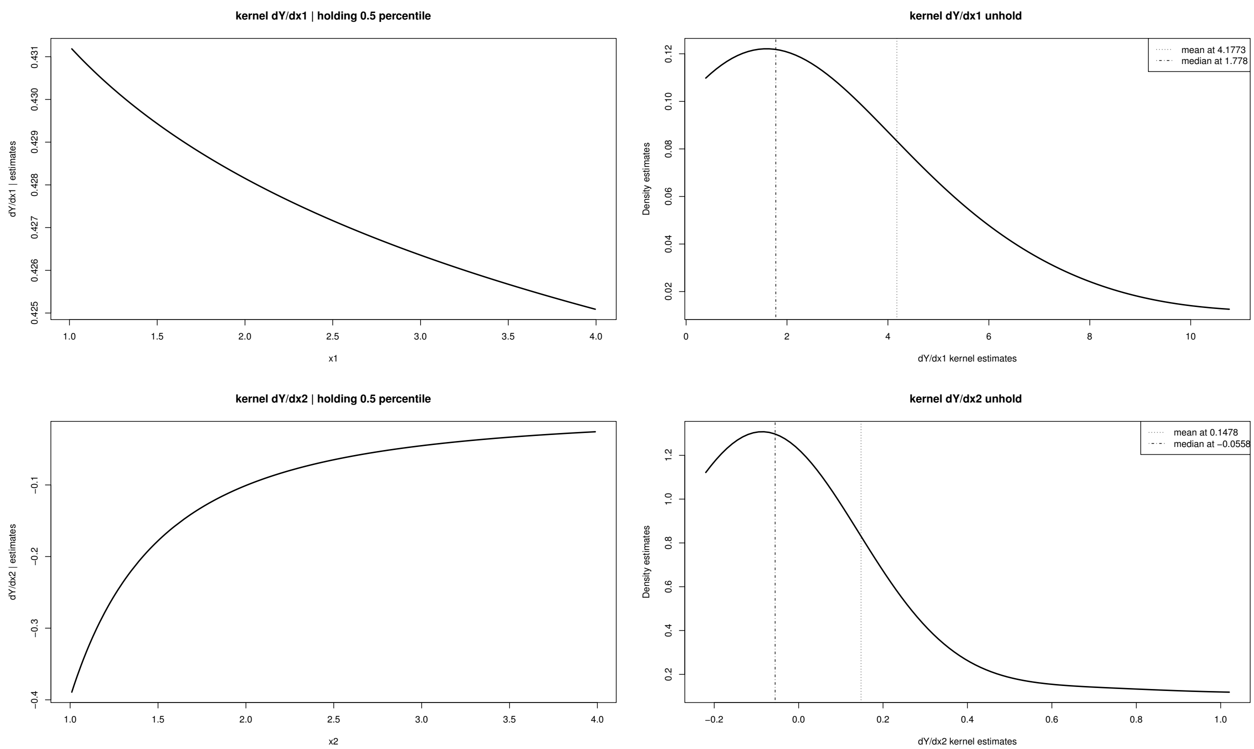


for computational efficiency to quickly obtain preliminary results without spending a much longer time for the kernel estimator (i.e., a local estimator with $p + 1$ backfitting steps). All additional arguments in WH.SCD() are listed with explanation in the script.

Function result.WH.SCD() in turn summarizes the estimation results from SCD.model. It automatically returns numerical results as a table, printed in the console panel of R. The first panel on the top summarizes the estimated parameters (by PNLS) in each coefficient function $\beta_j(\cdot)$. The second and third panel reports the summary for the coefficient function estimates (and related results) by series ($\hat{\beta}_j(\cdot)$) and by kernel ($\check{\beta}_j(\cdot)$), respectively. The fourth panel displays data information, tuning parameters, kernel function type, and nonparametric R-square ($corr(Y_{it}, \tilde{Y}_{it})$, with $\tilde{Y}_{it}$ being the fitted S-CD-FE by either series or kernel estimation).

Function result.WH.SCD() also prints out figures by default (figure=TRUE). If Kernel=TRUE, four figures will be printed, where the first two are related to numerical results in panel B of the table (series estimation), and the last two in panel C of the table (kernel estimation). If Kernel=FALSE, only the first two figures are displayed. Each figure will be displayed one at a time, and the next figure can be seen by hitting <Enter>. In the current example, the last two figures corresponding to panel C in result.scd are shown in Figure A2 and A3, respectively. Here, Figure A2 plots the three kernel estimated coefficient functions $\check{\beta}_j(\cdot)$ (red solid line) against $\{Z_{it,j}^\top \hat{\gamma}_0\}_{i=1,t=1}^{n,T}$ in the first, second, and third column, respectively, with point-wise 95% bootstrap confidence interval (black

Figure A3: Coefficients of Inputs Summary in the Package



dash line). The size of Figure A2 can be adjusted by the argument margin.b, which is currently margin.b=c(1,3). Figure A3 contains graphs with dimension $p \times 2$ ($p = 2$ in this case). The first column plots the kernel estimated partial effect of each input, say,

$$dY/dX_1 = \check{\beta}_0(Z_{it,0}^{\top}\gamma_0)\check{\beta}_1(Z_{it,1}^{\top}\gamma_1)\left(X_{it,1}^{\check{\beta}_1(Z_{it,1}^{\top}\gamma_1)-1}X_{it,2}^{\check{\beta}_2(Z_{it,2}^{\top}\gamma_2)}\right),$$

against X_1 in the first row, and dY/dX_2 against X_2 in the second row. Notice that in the first column of Figure A3, dY/dX_s is computed by holding all other variables at their respective medians (i.e., X_2 is held at its 50^{th} percentile in this case). Thus, the second column of Figure A3 graphs the estimated (kernel) density for dY/dX_s without holding all else at a certain level. Graphical results for series estimation are given in a similar fashion and thus not reported. Finally, the estimated return-to-scale (i.e., $\sum_{s=1}^{p} dY/dX_s$) is summarized on the bottom of panel B and C in the table returned by result.scd. All other arguments in result.WH.SCD() are given with explanations in the script. For further questions regarding the implementation of the package, please contact the first author via email.

References

Guo, C., Yang, H., Lv, J., 2016. Generalized varying index coefficient models. *Journal of Computational and Applied Mathematics*, 300, 1–17.