Economics 671: Econometrics
Department of Economics, Finance and Legal Studies
University of Alabama
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Midterm – Answers

1. (a) When ρ is different from unity, the problem is nonlinear in its parameters and cannot be estimated by OLS (in levels or logs).

   (b)
$$Q_n(\theta) = -\frac{1}{2n}\sum_{i=1}^{n}\left[y_i - \left(\beta_1 x_{1i}^{\rho} + \beta_2 x_{2i}^{\rho}\right)^{1/\rho}\right]^2$$

$$\begin{aligned}
\frac{\partial Q_n(\theta)}{\partial \beta_1} &= \frac{1}{n\rho}\sum_{i=1}^{n}\left(\beta_1 x_{1i}^{\rho} + \beta_2 x_{2i}^{\rho}\right)^{(1/\rho)-1}\left(x_{1i}^{\rho}\right)\left[y_i - \left(\beta_1 x_{1i}^{\rho} + \beta_2 x_{2i}^{\rho}\right)^{1/\rho}\right] \\
\frac{\partial Q_n(\theta)}{\partial \beta_2} &= \frac{1}{n\rho}\sum_{i=1}^{n}\left(\beta_1 x_{1i}^{\rho} + \beta_2 x_{2i}^{\rho}\right)^{(1/\rho)-1}\left(x_{2i}^{\rho}\right)\left[y_i - \left(\beta_1 x_{1i}^{\rho} + \beta_2 x_{2i}^{\rho}\right)^{1/\rho}\right]
\end{aligned}$$

$$V\left(\widehat{\beta}_{NLS}\right) = \left(\widehat{D}'\widehat{D}\right)^{-1}\widehat{D}'\widehat{\Omega}\widehat{D}\left(\widehat{D}'\widehat{D}\right)^{-1}$$

   where $\widehat{D} = \frac{\partial g(x,\beta)}{\partial \beta}|_{\beta=\widehat{\beta}}$ and $\widehat{\Omega} = diag\left(\widehat{u}_i^2\right)$.

   (c) Assuming normality of the errors

$$Q_n(\theta) = -\frac{n}{2}\ln\sigma^2 - \frac{n}{2}\ln 2\pi - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[y_i - \left(\beta_1 x_{1i}^{\rho} + \beta_2 x_{2i}^{\rho}\right)^{1/\rho}\right]^2$$

$$\begin{aligned}
\frac{\partial Q_n(\theta)}{\partial \beta_1} &= \sum_{i=1}^{n}\left(\beta_1 x_{1i}^{\rho} + \beta_2 x_{2i}^{\rho}\right)^{(1/\rho)-1}\left(x_{1i}^{\rho}\right)\left[y_i - \left(\beta_1 x_{1i}^{\rho} + \beta_2 x_{2i}^{\rho}\right)^{1/\rho}\right] \\
\frac{\partial Q_n(\theta)}{\partial \beta_2} &= \sum_{i=1}^{n}\left(\beta_1 x_{1i}^{\rho} + \beta_2 x_{2i}^{\rho}\right)^{(1/\rho)-1}\left(x_{2i}^{\rho}\right)\left[y_i - \left(\beta_1 x_{1i}^{\rho} + \beta_2 x_{2i}^{\rho}\right)^{1/\rho}\right]
\end{aligned}$$

$$V\left(\widehat{\beta}_{MLE}\right) = -\left[E\left(\frac{\partial^2 Q_n(\theta)}{\partial\theta\partial\theta'}\right)^{-1}\right]$$

   (d) If we assume strict exogeneity of the inputs, we can set $h(w,\theta) = x'[y - g(x,\beta)]$ and further letting $W_n = I$

$$\begin{aligned}
Q_n(\theta) = &\left[\frac{1}{n}\sum_{i=1}^{n}\left[y_i - \left(\beta_1 x_{1i}^{\rho} + \beta_2 x_{2i}^{\rho}\right)^{1/\rho}\right]'\left(x_{1i}\ x_{2i}\right)\right]' \\
&\left[\frac{1}{n}\sum_{i=1}^{n}\left(x_{1i}\ x_{2i}\right)'\left[y_i - \left(\beta_1 x_{1i}^{\rho} + \beta_2 x_{2i}^{\rho}\right)^{1/\rho}\right]\right]
\end{aligned}$$

$$\frac{\partial Q_n(\theta)}{\partial \beta_1} = \left[\frac{1}{n}\sum_{i=1}^{n}\left[\left(\beta_1 x_{1i}^{\rho}+\beta_2 x_{2i}^{\rho}\right)^{(1/\rho)-1}\left(x_{1i}^{\rho}\right)\right]'\left(x_{1i}\ x_{2i}\right)\right]'$$
$$\left[\frac{1}{n}\sum_{i=1}^{n}\left(x_{1i}\ x_{2i}\right)'\left[y_i-\left(\beta_1 x_{1i}^{\rho}+\beta_2 x_{2i}^{\rho}\right)^{1/\rho}\right]\right]$$
$$\frac{\partial Q_n(\theta)}{\partial \beta_2} = \left[\frac{1}{n}\sum_{i=1}^{n}\left[\left(\beta_1 x_{1i}^{\rho}+\beta_2 x_{2i}^{\rho}\right)^{(1/\rho)-1}\left(x_{2i}^{\rho}\right)\right]'\left(x_{1i}\ x_{2i}\right)\right]'$$
$$= \left[\frac{1}{n}\sum_{i=1}^{n}\left(x_{1i}\ x_{2i}\right)'\left[y_i-\left(\beta_1 x_{1i}^{\rho}+\beta_2 x_{2i}^{\rho}\right)^{1/\rho}\right]\right]$$

$$V\left(\widehat{\beta}_{GMM}\right) = \left(\widehat{G}'\widehat{G}\right)^{-1}\widehat{G}'\widehat{S}\widehat{G}\left(\widehat{G}'\widehat{G}\right)^{-1}$$

where $\widehat{G} = \frac{\partial h(w,\theta)}{\partial \beta}|_{\beta=\widehat{\beta}}$ and $\widehat{S} = \frac{1}{n}\sum_{i=1}^{n} h\left(w_i,\widehat{\theta}\right) h\left(w_i,\widehat{\theta}\right)'$.

2. (a) No.

   (b) $\ln L(\theta) = \frac{1}{n}\sum_{i=1}^{n}\left[x_i\beta + \ln\alpha + (\alpha-1)\ln y_i - \exp(x_i\beta)\, y_i^{\alpha}\right]$

   (c) $\frac{\partial \ln L(\theta)}{\partial \beta} = \frac{1}{n}\sum_{i=1}^{n}\left[1-\exp(x_i\beta)\, y_i^{\alpha}\right] x_i = 0,$

   $\frac{\partial \ln L(\theta)}{\partial \alpha} = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\alpha} + \ln y_i - \exp(x_i\beta)\, y_i^{\alpha}\ln y_i\right] = 0$

3. (a) $\widetilde{r}_{it} = \alpha_i + \beta_i \widetilde{r}_{pt} + \widetilde{\varepsilon}_{it}$, where $\widetilde{r}_{it}$ is the excess return on asset i in time period t, $\widetilde{r}_{pt}$ is the excess return on portfolio p in time period t, α_i and β_i are the asset specific intercept and slope parameters, respectively and $\widetilde{\varepsilon}_{it}$ is the disturbance term

   (b) $E(\widetilde{\varepsilon}_{it}) = 0$ and $E(\widetilde{\varepsilon}_{it}\widetilde{r}_{pt}) = 0$ which are simply the normal equations and estimation via GMM in this just identified case is a MM estimator and equivalent to OLS

   (c) The conditions above plus $\alpha_i = 0\ \forall i$ and this leads to an over identified model and hence GMM estimation

   (d) If the errors are heteroskedastic or autocorrelated and the maximum likelihood estimator does not take those into account, then both estimation and inference will be problematic (biases and improper inference)