Economics 670: Econometrics

Department of Economics, Finance and Legal Studies

University of Alabama

Fall 2021

Midterm II - Key

The exam consists of three questions on three pages. Each question is of equal value.

1. Let $\mu_r = E(Y^r)$ be the rth moment of the random variable $Y \sim (0, \sigma^2)$, where we assume that $E(Y^{2r}) < \infty$. With this information:

   (a) Write down the natural estimator for $\widehat{\mu}_r$ of μ_r.

   (b) Using large sample theory, show that the estimator from part (a) is consistent.

   (c) Find the asymptotic distribution of $\sqrt{n}(\widehat{\mu}_r - \mu_r)$ as the sample size $n \to \infty$.

(a) $\hat{\mu}_r = \frac{1}{n} \sum_{i=1}^{n} y_i^r$

(b) by WLLN

$\frac{1}{n} \sum_{i=1}^{n} y_i^r \xrightarrow{p} E(y^r) = \mu_r$

$\Rightarrow \hat{\mu}_r \xrightarrow{p} \mu$ & is thus consistent

(c) $E(\hat{\mu}_r) = E\left(\frac{1}{n} \sum_{i=1}^{n} y_i^r\right) = \mu_r$

$V(\hat{\mu}_r) = E\left[(\hat{\mu}_r - \mu_r)^2\right] = E(\hat{\mu}_r^2) - E(\mu_r)^2$

$E(\hat{\mu}_r^2) = E\left(\frac{1}{n} \sum_{i=1}^{n} y_i^{2r}\right) = \mu_{2r}$

$V(\hat{\mu}_r) = \mu_{2r} - \mu_r^2$

$\sqrt{n}(\hat{\mu}_r - \mu_r) \xrightarrow{d} N(0, (\mu_{2r} - \mu_r^2))$

2. Consider the model $y = X\beta + e$, for a random sample of $i = 1, 2, \ldots, n$ observations with k regressors, where $e|X \sim N(0, \sigma^2)$. For this model,

   (a) Derive the maximum likelihood estimator of σ^2

   (b) Derive the variance of the estimator in part (a)

   (c) Using large sample theory, show that the estimator from part (a) is consistent

(a) $\ln \mathcal{L}(e) = -\frac{n}{2} \ln 2\pi - \frac{n}{2} \ln \sigma^2 - \frac{1}{2\sigma^2} e'e$

$$\frac{\partial \ln \mathcal{L}(e)}{\partial \sigma^2} = \frac{-n}{2\hat{\sigma}^2} + \frac{1}{2(\hat{\sigma}^2)^2} \hat{e}'\hat{e} = 0$$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{n} \hat{e}'\hat{e}$$

(b) $$\frac{\partial^2 \ln \mathcal{L}(e)}{\partial (\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{e'e}{\sigma^6}$$

note $\sigma^2 = \frac{1}{n} e'e$

CRLB is $-E[I]^{-1}$

$$-E\left[\frac{n}{2\sigma^4} - \frac{e'e}{\sigma^6}\right] = -E\left[\frac{n}{2\sigma^4} - \frac{n\sigma^2}{\sigma^6}\right]$$

$$= -E\left[\frac{n}{2\sigma^4} - \frac{2n\sigma^2}{2\sigma^6}\right]$$

$$= -E\left[-\frac{n}{2\sigma^4}\right]$$

$$= \frac{2\sigma^4}{n}$$

(c) $\hat{\sigma}^2 = \frac{1}{n}\hat{e}'\hat{e}$

$$= \frac{1}{n}(y - x\hat{\beta})'(y - x\hat{\beta})$$

$$= \frac{1}{n}(e + x\beta - x\hat{\beta})'(e + x\beta - x\hat{\beta})$$

$$= \frac{1}{n}e'e - 2\cdot\frac{1}{n}x'e(\hat{\beta} - \beta)$$

$$+ (\hat{\beta} - \beta)'\frac{1}{n}x'x(\hat{\beta} - \beta)$$

WLLN shows

$$\frac{1}{n}e'e \xrightarrow{p} \sigma^2$$

$$\frac{1}{n}x'e \xrightarrow{p} E(xe) = 0$$

$$\frac{1}{n}x'x \xrightarrow{p} E(xx')$$

using $\hat{\beta} \xrightarrow{p} \beta$ we know that

$$\hat{\sigma}^2 \xrightarrow{p} \sigma^2$$

3. Consider the model $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + e$, where $e|x_1, x_2 \sim N(0, \sigma^2)$ where σ^2 is known and x_1 and x_2 are scalars. Suppose we are interested in testing the null $H_0 : \beta_2 = 0$.

(a) Write down the test statistic for a t-test. What is the asymptotic distribution of this statistic under the null?

(b) Write down the test statistic for a likelihood-ratio test. What is the asymptotic distribution of this statistic under the null?

(c) Write down the test statistic for an F-test. What is the asymptotic distribution of this statistic under the null?

(a) $$t = \frac{\hat{\beta}_2 - 0}{\sqrt{V(\hat{\beta}_2)}} \sim N(0,1)$$

(b) $$LR = -2\ln\left(\frac{\mathcal{L}(\tilde{\beta}, \sigma^2)}{\mathcal{L}(\hat{\beta}, \sigma^2)}\right) \sim \chi^2_1$$

(c) $$F = \frac{(SSR_R - SSR_u)/1}{SSR_u/(n-3)} \sim F_{1,(n-3)}$$

R: $y = \beta_0 + \beta_1 x_1 + u$

$\ln \mathcal{L}(\tilde{\beta}, \sigma^2)$

$SSR_R = \sum_{i=1}^{n} \tilde{u}_i^2$

u: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + e$

$\ln \mathcal{L}(\hat{\beta}, \sigma^2)$

$SSR_u = \sum_{i=1}^{n} \hat{u}_i^2$