Economics 670: Econometrics

Department of Economics, Finance and Legal Studies

University of Alabama

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Midterm II Key

The exam consists of three questions on three pages. Each question is of equal value.

1. Consider the regression model $y = X\beta + e$, where we assume that $E(e|X) = 0$. For each scenario below, show that a given estimator of β is asymptotically normal. Be sure to specify the mean and variance of the estimator (or its normalized version).
   (a) Using the assumption e is i.i.d. $N(0, \sigma^2)$
   (b) Using large sample theory (be sure to note assumptions/laws being used)

(a) $\hat{\beta} = (X'X)^{-1}X'y = (X'X)^{-1}X'(X\beta + e) = \beta + (X'X)^{-1}X'e$

$\hat{\beta} - \beta|_X \sim (X'X)^{-1}X' N(0, \sigma^2 I_n)$

$\sim N(0, \sigma^2 (X'X)^{-1} X' I_n X (X'X)^{-1})$

$= N(0, \sigma^2 (X'X)^{-1})$

(b) $\hat{\beta} = \left(\frac{1}{n}\sum_{i=1}^n x_i x_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n x_i y_i\right) = \left(\frac{1}{n}\sum_{i=1}^n x_i x_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n x_i (x_i'\beta + e_i)\right) = \beta + \left(\frac{1}{n}\sum_{i=1}^n x_i x_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n x_i e_i\right)$

by WLLN $\frac{1}{n}\sum_{i=1}^n x_i x_i' \xrightarrow{p} E(x_i x_i')$

by WLLN $\frac{1}{n}\sum_{i=1}^n x_i e_i \xrightarrow{p} E(x_i e_i) = 0$

$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n}\sum_{i=1}^n x_i x_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i e_i\right)$

assuming the variance of $x_i e_i$ is finite

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i e_i \xrightarrow{d} N(0, \Omega)$$ as $n \to \infty$ (CLT)

where $\Omega = E(x_i x_i' e_i^2) < \infty$

then we can show that

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} E(x_i x_i')^{-1} N(0, \Omega)$$

$$= N(0, E(x_i x_i')^{-1} \Omega E(x_i x_i')^{-1})$$

$$= N(0, \sigma^2 E(x_i x_i')^{-1})$$

2. Consider the regression model $y = X\beta + e$, where $e \sim N(0, \sigma^2)$. Suppose we have a consistent, asymptotically normal estimator $\widehat{\beta}$ of β whereby β is a scalar and we have a consistent, asymptotically normal estimator s^2 of σ^2. Consider the statistic

$$T = \frac{\widehat{\beta} - \beta}{se\left(\widehat{\beta}\right)},$$

where $se(\widehat{\beta})$ is the standard error of the estimator (i.e., square root of the estimated variance of $\widehat{\beta}$). With this information, answer the following (show your work):

(a) What is the finite sample distribution of this statistic?

(b) What is the asymptotic distribution of this statistic?

(a) $$T = \frac{\hat{\beta} - \beta}{se(\hat{\beta})} = \frac{\hat{\beta} - \beta}{\sqrt{s^2 (X'X)^{-1}}} = \frac{\dfrac{\hat{\beta} - \beta}{\sqrt{\sigma^2 (X'X)^{-1}}}}{\sqrt{\dfrac{(n-k)s^2}{\sigma^2} / (n-k)}}$$

$$\sim \frac{N(0,1)}{\sqrt{\chi^2_{n-k} / (n-k)}} \sim t_{n-k}$$

(b) $$T = \frac{\hat{\beta} - \beta}{se(\hat{\beta})} = \frac{\sqrt{n}(\hat{\beta} - \beta)}{\sqrt{\hat{V}_\beta}} \xrightarrow{d} \frac{N(0,1)}{\sqrt{V_\beta}}$$

$$= N(0,1)$$

because $\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, V_\beta)$ by CLT

and $\hat{V}_\beta \xrightarrow{p} V_\beta$

3. Consider the R code that we discussed in lecture. Next to each line of code, briefly comment on what that line of code is doing.

n < − 100	Sample size
b < − 1000	# of simulations
alpha < − numeric(b)	Storage for $\hat{\alpha}, \hat{\beta}, \sqrt{n}(\hat{\alpha}-\alpha), \sqrt{n}(\hat{\beta}-\beta)$
beta < − numeric(b)	
clt.alpha < − numeric(b)	
clt.beta < − numeric(b)	
for (j in 1:b){	loop over the # of simulations
u < − runif(n,-0.5,0.5)	$U[-\frac{1}{2}, \frac{1}{2}]$ of size n
x < − rnorm(n,0,1)	$N(0,1)$ of size n
y < − 1 + x + u	$y = \alpha + \beta x + u = 1 + x + u$
ols.estimates < − lm(y~x)	OLS regression of y on x
alpha[j] < − ols.estimates$coef[1]	$\hat{\alpha}$
beta[j] < − ols.estimates$coef[2]	$\hat{\beta}$
clt.alpha[j] < − sqrt(n)*(alpha[j]-1)	$\sqrt{n}(\hat{\alpha}-\alpha) = \sqrt{n}(\hat{\alpha}-1)$
clt.beta[j] < − sqrt(n)*(beta[j]-1)	$\sqrt{n}(\hat{\beta}-\beta) = \sqrt{n}(\hat{\beta}-1)$
}	end of loop
hist(clt.alpha)	dist of $\sqrt{n}(\hat{\alpha}-\alpha)$
sd(clt.alpha)∧2	variance of $\sqrt{n}(\hat{\alpha}-\alpha)$
hist(clt.beta)	dist of $\sqrt{n}(\hat{\beta}-\beta)$
sd(clt.beta)∧2	variance of $\sqrt{n}(\hat{\beta}-\beta)$