Economics 670: Econometrics

Department of Economics, Finance and Legal Studies

University of Alabama

Fall 2019

Midterm II - Key

The exam consists of three questions on three pages. Each question is of equal value.

1. Consider the model $y = X\beta + e$, for a random sample of $i = 1, 2, \ldots, n$ observations and k regressors, where e is normally distributed with mean 0 and variance σ^2. For this model,

   (a) Derive the OLS estimator of β

   (b) Derive the variance of the estimator in part (a)

   (c) Derive the MLE estimator of β

   (d) Derive the variance of the estimator in part (c)

(a) $\arg\min_{\beta} \hat{u}'\hat{u} = (y - X\hat{\beta})'(y - X\hat{\beta})$

$\frac{\partial}{\partial \beta} = X'y - X'X\hat{\beta} = 0 \Rightarrow \hat{\beta}_{OLS} = (X'X)^{-1}X'y$

(b) $V(\hat{\beta}|X) = E\{[\hat{\beta} - E(\hat{\beta}|X)][\hat{\beta} - E(\hat{\beta}|X)]' | X]$

$= E\{(X'X)^{-1}X'ee'X(X'X)^{-1} | X]$

$= (X'X)^{-1}X'\sigma^2 I X (X'X)^{-1}$

$= \sigma^2 (X'X)^{-1}$

(c) $\ln \mathcal{L}(\beta, \sigma^2) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\hat{u}'\hat{u}$

$\frac{\partial}{\partial \beta}$ $\frac{1}{\sigma^2}(X'y - X'X\hat{\beta}) = 0$

$\Rightarrow \hat{\beta}_{MLE} = (X'X)^{-1}X'y$

(d) $\frac{\partial^2}{\partial\beta\partial\beta'} = \frac{1}{\sigma^2}(X'X)$

$\Rightarrow$

$V(\hat{\beta}_{MLE}) = \sigma^2(X'X)^{-1}$

2. Consider the random variable y which has mean zero and variance σ^2. Define the third moment of y by $\mu_3 = E(y^3)$. For a random sample of size n,

   (a) Construct an estimator $\widehat{\mu}_3$ for μ_3.

   (b) Show that $\widehat{\mu}_3$ is an unbiased estimator for μ_3.

   (c) Calculate the variance of $\widehat{\mu}_3$, say $V(\widehat{\mu}_3)$.

(a) $\hat{\mu}_3 = \frac{1}{n} \sum_{i=1}^{n} y_i^3$

(b) $E(\hat{\mu}_3) = E\left(\frac{1}{n} \sum_{i=1}^{n} y_i^3\right)$

$= \frac{1}{n} \sum_{i=1}^{n} E(y_i^3)$

$= \frac{1}{n} \sum_{i=1}^{n} \mu_3$

$= \mu_3$

(c) $V(\hat{\mu}_3) = V\left(\frac{1}{n} \sum_{i=1}^{n} y_i^3\right)$

$V(\hat{\mu}_3) = E\{[\hat{\mu}_3 - E(\hat{\mu}_3)]^2\}$

$= E[\hat{\mu}_3^2 - E(\hat{\mu}_3)^2]$

$= \frac{1}{n}(\mu_6 - \mu_3^2)$

* if $N(0,1)$

$V(\hat{\mu}_3) = n^{-1}[E(y^6) - E(y^3)^2]$

$= n^{-1}[5!! - (3!!)^2]$

$= \frac{6}{n}$

3. Consider the R code below. Next to each line of code, briefly comment on what that line of code is doing.

`## R code for Question 3 - MT2` comment

`rm(list=ls())` clear memory

`set.seed(123456)` set seed

`n <- 100` sample size

`x <- rnorm(n,0,1)` $n\times 1$ regressor $N(0,1)$

`u <- rnorm(n,0,0.1)` $n\times 1$ error $N(0, 0.1)$

`y <- 1 + 0.5*x + u` generate $y = 1 + \frac{1}{2}x + u$

`x <- as.matrix(x)` x is a matrix

`y <- as.matrix(y)` y is a matrix

`xm <- mean(x)` $\bar{x}$

`ym <- mean(y)` $\bar{y}$

`ones <- as.matrix(rep(1,n))` column of ones

`X <- cbind(ones,x)` $X = (1\ x)$

`b <- solve(t(X)%*%X)%*%(t(X)%*%y)` $\hat{\beta} = (x'x)^{-1}x'y$

`u <- y - X%*%b` residual $\hat{u} = y - x\hat{\beta}$

`yh <- X%*%b` $\hat{y} = x\hat{\beta}$

`st <- sumc((y-ym)^2)` $SST = \sum (y_i - \bar{y})^2$

`se <- sumc((yh-ym)^2)` $SSE = \sum (\hat{y}_i - \bar{y})^2$

`sr <- sumc((y-yh)^2)` $SSR = \sum \hat{u}_i^2$

`r2 <- 1-sr/st` $R^2 = 1 - SSR/SST$

`s2 <- solve(n-1)*sumc(u^2)` $\hat{\sigma}^2 = \frac{1}{n-k}\sum \hat{u}_i^2$

`vb <- [illegible]s2*solve(t(X)%*%X)` $\hat{\sigma}^2(x'x)^{-1}$

`t <- (b-0)/sqrt(vb[2,2])`

$H_0: b = 0$

$H_1: b \neq 0$

$$t = \frac{\hat{\beta} - 0}{se(\hat{\beta})}$$