Economics 670: Econometrics

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University of Alabama

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Midterm I Key

The exam consists of three questions on three pages. Each question is of equal value.

1. Consider the model $y = x'\beta + e$, where $E(e|x) = 0$ and $E(e^2|x) = \sigma^2(x)$. With this information, answer the following:

   (a) Show the conditional expectation of y given x (i.e., $E(y|x)$).

   (b) Show that $var(y|x) = \sigma^2(x)$.

   (c) By using the law of iterative expectations, show that the unconditional error variance is equal to the average conditional variance (i.e., $\sigma^2 = E(\sigma^2(x))$).

   (d) Suppose we re-scale the error via $\epsilon = \frac{e}{\sigma(x)}$. Show that the expectation of ϵ given x is zero (i.e., $E(\epsilon|x)$=0).

   (e) For the re-scaled error in part (d), show that the conditional variance of ϵ given x is equal to one (i.e., $var(\epsilon|x) = 1$).

(a) $E(y|x) = E(x'\beta + e|x) = E(x'|x)\beta + E(e|x) = x'\beta$

(b) $V(y|x) = V(x'\beta + e|x) = V(x'|x)\beta + V(e|x) = \sigma^2(x)$

(c) $\sigma^2 = E(e^2) = E[E(e^2|x)] = E[\sigma^2(x)]$

(d) $E(\epsilon|x) = E\left(\frac{e}{\sigma(x)}\Big|x\right) = \frac{1}{\sigma(x)}E(e|x) = 0$

(e) $V(\epsilon|x) = E(\epsilon^2|x) = E\left[\frac{e^2}{\sigma^2(x)}\Big|x\right] = \frac{1}{\sigma^2(x)}E(e^2|x) = \frac{\sigma^2(x)}{\sigma^2(x)} = 1$

2. Consider a random sample $\{x_i, y_i\}_{i=1}^n$ from the data generating process $y_i = x_i'\beta + e_i$, where $E(e_i|x_i) = 0$. With this information, answer the following (you may use matrix notation if desired):

   (a) Defining the sum of squared errors as $SSE(\beta) = \sum_{i=1}^n (y_i - x_i'\beta)^2$, take the first-order-condition to solve for the estimator for β.

   (b) Exploiting the objective function from part (a), show that this results in a minimum (i.e., second derivative).

   (c) Show that the estimator from part (a) is an unbiased estimator of β.

   (d) Suppose we have a heteroskedastic error, derive the conditional variance of the estimator from part (a).

   (e) Suppose we have a homoskedastic error, simplify your answer from part (d).

(a) $SSE(\beta) = \sum_{i=1}^n y_i^2 - 2\beta' \sum_{i=1}^n x_i y_i + \beta' \sum_{i=1}^n x_i x_i' \beta$

$$\frac{\partial SSE(\hat\beta)}{\partial \beta} = -2\sum_{i=1}^n x_i y_i + 2\sum_{i=1}^n x_i x_i' \hat\beta = 0$$

$$\sum_{i=1}^n x_i x_i' \hat\beta = \sum_{i=1}^n x_i y_i$$

$$\hat\beta = \left(\sum_{i=1}^n x_i x_i'\right)^{-1}\left(\sum_{i=1}^n x_i y_i\right)$$

(b) $$\frac{\partial^2 SSE(\hat\beta)}{\partial\beta\,\partial\beta'} = 2\sum_{i=1}^n x_i x_i' > 0$$

is pd so it is the unique min of $SSE(\beta)$

(c) $$E(\hat\beta|x) = E\left[\left(\sum_{i=1}^n x_i x_i'\right)^{-1}\left(\sum_{i=1}^n x_i y_i\right)\Big|x\right]$$

$$= E\left\{\left[\sum_{i=1}^n x_i x_i'\right]^{-1}\left[\sum_{i=1}^n x_i (x_i'\beta + e_i)\right]\Big|x\right\}$$

$$= \beta + \left(\sum_{i=1}^n x_i x_i'\right)^{-1}\sum_{i=1}^n x_i E(e_i|x) = \beta + 0 = \beta$$

(d) $var(\hat{\beta}|x) = E\{[\hat{\beta}-\beta|x][\hat{\beta}-\beta|x]'\}$

$$= E\{[(\sum_{i=1}^{n} x_i x_i')^{-1}(\sum_{i=1}^{n} x_i e_i)|x]$$
$$[(\sum_{i=1}^{n} x_i x_i')^{-1}(\sum_{i=1}^{n} x_i e_i)|x]'\}$$

$$= (\sum_{i=1}^{n} x_i x_i')^{-1} \sum_{i=1}^{n} x_i x_i' \sigma_i^2 (\sum_{i=1}^{n} x_i x_i')^{-1}$$

(e) if $\sigma_i^2 = \sigma^2 \; \forall i$

$$var(\hat{\beta}|x) = (\sum_{i=1}^{n} x_i x_i')^{-1} \sigma^2 \sum_{i=1}^{n} x_i x_i' (\sum_{i=1}^{n} x_i x_i')^{-1}$$

$$= \sigma^2 (\sum_{i=1}^{n} x_i x_i')^{-1}$$

3. Consider the R code below. For this code, answer the following:

```
## R code for Question 3 - MT1

n <- 100

x <- rnorm(n,0,1)

mu1 <- mean(x)

mu2 <- mean(x^2)

mu3 <- mean(x^3)

mu4 <- mean(x^4)

mu2c <- mean((x-mu1)^2)

mu3c <- mean((x-mu1)^3)

mu4c <- mean((x-mu1)^4)
```

(a) What are mu1, mu2, mu3 and mu4 estimating? Write the equation for each.

(b) Pick one of the estimators (mu1, mu2, mu3, or mu4) from part (a) and show that it is an unbiased estimator of the true underlying population parameter.

(c) What are mu2c, mu3 and mu4c estimating? Write the equation for each.

(d) What is the common name for each value in part (c)?

(e) What do we expect the value of each to be (from part c)?

(a) first for uncentered moments

$\hat{\mu}_1 = \frac{1}{n} \sum_{i=1}^{n} x_i$ $\qquad$ $\hat{\mu}_3 = \frac{1}{n} \sum_{i=1}^{n} x_i^3$

$\hat{\mu}_2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2$ $\qquad$ $\hat{\mu}_4 = \frac{1}{n} \sum_{i=1}^{n} x_i^4$

(b) $E(\hat{\mu}_1) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i)$

$= \frac{1}{n}\sum_{i=1}^{n}\mu_1 = \frac{1}{n} n\mu_1 = \mu_1$

(c) variance, skewness & kurtosis

$\hat{\mu}_{2c} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$

$\hat{\mu}_{3c} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^3$

$\hat{\mu}_{4c} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^4$

central moments

(d) variance, skewness, kurtosis

(e) $x \sim N(0,1)$

mean $= 0$

variance $= 1$

skewness $= 0$

kurtosis $= 3$