Economics 670: Econometrics

Department of Economics, Finance and Legal Studies

University of Alabama

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Midterm I — Answer Key

The exam consists of three questions on three pages. Each question is of equal value.

1. Consider the random variable X which follows a uniform distribution with support $x \in [a, b]$. Noting that the probability density function of a uniform random variable is $f_X(x) = \frac{1}{b-a}$ for $x \in [a, b]$ and zero otherwise, derive the first two centered moments (i.e., mean and variance). Without deriving the result, what is the value for third centered moment of X (i.e., skewness)?

$$E(x) = \int_a^b \frac{x}{b-a}\,dx = \frac{x^2}{2(b-a)}\Big|_a^b = \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{(b+a)(b-a)}{2(b-a)} = \boxed{\frac{a+b}{2}}$$

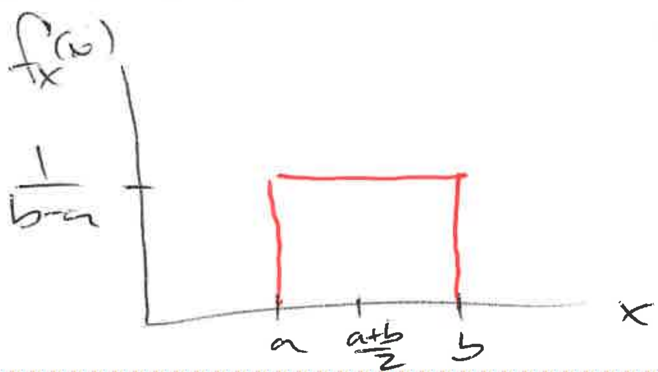
$$E(x^2) = \int_a^b \frac{x^2}{b-a}\,dx = \frac{x^3}{3(b-a)}\Big|_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$V(x) = E(x^2) - E(x)^2$$

$$= \frac{b^3-a^3}{3(b-a)} - \frac{(a+b)^2}{4} = \frac{a^2-2ab+b^2}{12}$$

$$= \boxed{\frac{(b-a)^2}{12}}$$

skewness of uniform dist is zero, graphically



mean = median = mode $= \frac{a+b}{2}$

2. Suppose we have data on 22 individuals, where $y = ln(income)$ and $x = ln(education)$:

$$\overline{y} = 20, \overline{x} = 10, \underbrace{\sum_{i=1}^{n}(y_i - \overline{y})^2}_{=100}, \sum_{i=1}^{n}(x_i - \overline{x})^2 = 60, \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) = 30$$

(a) Compute the linear projection estimates of α and β in the model $y = \alpha + x\beta + e$

(b) Compute SST, SSR, SSE, R^2 and $\widehat{\sigma}^2$

(a) $\hat{\beta} = \dfrac{\sum_{i=1}^{n}(y_i-\bar{y})(x_i-\bar{x})}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \dfrac{30}{60} = \boxed{\dfrac{1}{2}}$

$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} = 20 - \frac{1}{2}\cdot 10 = \boxed{15}$

(b) $SST = \sum_{i=1}^{n}(y_i-\bar{y})^2 = \boxed{100}$

$SSR = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 = \sum_{i=1}^{n}\left[(\hat{\alpha}+\hat{\beta}x_i) - (\hat{\alpha}+\hat{\beta}\bar{x})\right]^2$

$= \sum_{i=1}^{n}\hat{\beta}^2(x_i-\bar{x})^2 = \hat{\beta}^2\sum_{i=1}^{n}(x_i-\bar{x})^2 = \left(\frac{1}{2}\right)^2 60 = \boxed{15}$

$SSE = SST - SSR = 100 - 15 = \boxed{85}$

$R^2 = \dfrac{SSR}{SST} = \dfrac{85}{100} = 0.85$

$\hat{\sigma}^2 = \dfrac{1}{n-2} SSE = \dfrac{1}{22-2} 85 = \boxed{\dfrac{85}{20}}$

3. Let $\widehat{\beta}_1$ and $\widehat{\beta}_2$ denote the estimated regression coefficients for the linear model $y = X_1\beta_1 + X_2\beta_2 + e$ where X_1 is a $n \times k_1$ matrix, X_2 is a $n \times k_2$ matrix and β_1 and β_2 are of dimensions k_1 and k_2, respectively. Show that $\widetilde{\beta}_1 = \widehat{\beta}_1 + (X_1'X_1)^{-1}X_1'X_2\widehat{\beta}_2$, where $\widetilde{\beta}_1$ is a vector of coefficients from the regression of y solely on X_1. Argue/show that $\widehat{\beta}_1$ is an unbiased estimator of β_1. Under what conditions is $\widetilde{\beta}_1$ an unbiased estimator for β_1. Note that $\widehat{\beta}_1 = \widetilde{\beta}_1 - (X_1'X_1)^{-1}X_1'X_2\widehat{\beta}_2 = (X_1'X_1)^{-1}X_1'(y - X_2\widehat{\beta}_2)$.

$$y = X_1\beta_1 + X_2\beta_2 + e$$

$$= X\beta + e$$

$$\hat{\beta} = (X'X)^{-1}X'y$$

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix}^{-1} \begin{bmatrix} X_1'y \\ X_2'y \end{bmatrix}$$

$$\begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1'y \\ X_2'y \end{bmatrix}$$

$$X_1'X_1\hat{\beta}_1 + X_1'X_2\hat{\beta}_2 = X_1'y$$

$$\hat{\beta}_1 = (X_1'X_1)^{-1}(X_1'y) - (X_1'X_1)^{-1}(X_1'X_2)\hat{\beta}_2$$

$$= (X_1'X_1)^{-1}X_1'(y - X_2\hat{\beta}_2)$$

$$= \tilde{\beta}_1 - (X_1'X_1)^{-1}X_1'X_2\hat{\beta}_2$$

$$\tilde{\beta}_1 = \hat{\beta}_1 + (X_1'X_1)^{-1}X_1'X_2\hat{\beta}_2$$

$$y = X_1\beta_1 + e_1$$

$$\tilde{\beta}_1 = (X_1'X_1)^{-1}X_1'y$$

$$= (X_1'X_1)^{-1}X_1'(X_1\beta_1 + X_2\beta_2 + e)$$

$$= (X_1'X_1)^{-1}X_1'X_1\beta_1 + (X_1'X_1)^{-1}X_1'X_2\beta_2$$

$$+ (X_1'X_1)^{-1}X_1'e$$

$$= \beta_1 + (X_1'X_1)^{-1}X_1'X_2\beta_2 + (X_1'X_1)^{-1}X_1'e$$

$$E(\tilde{\beta}_1) = \beta_1 + (X_1'X_1)^{-1}X_1'X_2\beta_2$$

$\neq \beta_1$ unless $\beta_2 = 0$ or

$X_1'X_2 = 0$