Economics 670: Econometrics

Department of Economics, Finance and Legal Studies

University of Alabama

Fall 2021

Midterm I – Key

The exam consists of three questions on three pages. Each question is of equal value.

1. Consider the model $y = X\beta + e$, where $E(e|X) = 0$ and $E(e^2|X) = \Omega$, where Ω is a known variance-covariance matrix. Our goal is to estimate β via minizing the objective function $\tilde{e}'\Omega^{-1}\tilde{e}$, where $\tilde{e} = y - X\tilde{\beta}$. With this information

   (a) Derive the estimator $\tilde{\beta}$ of β.

   (b) Show that the estimator from part (a) is unbiased.

   (c) Derive the variance of the estimator from part (a).

   (d) Show that the variance from part (c) is smaller than that of the variance of the OLS estimator.

(a) $(y - X\tilde{\beta})'\Omega^{-1}(y - X\tilde{\beta})$

$$\frac{\partial}{\partial\beta} = -2X'\Omega^{-1}(y - X\tilde{\beta}) = 0$$

$$X'\Omega^{-1}y - X'\Omega^{-1}X\tilde{\beta} = 0$$

$$\tilde{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$$

(b) $\tilde{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}(X\beta + e)$

$$= \beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}e$$

$$E(\tilde{\beta} - \beta|X) = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}E(e|X)$$

$$= 0$$

(c) $V(\tilde{\beta} - \beta | x) = V\left[(x'\Omega^{-1}x)^{-1} x'\Omega^{-1} e | x \right]$

$= (x'\Omega^{-1}x)^{-1} x'\Omega^{-1} V(e|x) \Omega^{-1} x (x'\Omega^{-1}x)^{-1}$

$= (x'\Omega^{-1}x)^{-1} x'\Omega^{-1} \Omega \Omega^{-1} x (x'\Omega^{-1}x)^{-1}$

$= (x'\Omega^{-1}x)^{-1} x'\Omega^{-1} x (x'\Omega^{-1}x)^{-1}$

$= (x'\Omega^{-1}x)^{-1}$

(d) $\hat{\beta}_{OLS} = (x'x)^{-1} x'y$

$\hat{\beta} - \beta = (x'x)^{-1} x'e$

$V(\hat{\beta} - \beta | x) = V\left[(x'x)^{-1} x'e | x \right]$

$= (x'x)^{-1} x' V(e|x) x (x'x)^{-1}$

$= (x'x)^{-1} x'\Omega x (x'x)^{-1}$

$V(\hat{\beta}_{OLS} - \beta | x) - V(\tilde{\beta} - \beta | x) \geq 0$?

$(x'x)^{-1} x'\Omega x (x'x)^{-1} - (x'\Omega^{-1}x)^{-1}$

$= (x'x)^{-1} x'\Omega x (x'x)^{-1} - (x'\Omega^{-1}x)^{-1} x'\Omega^{-1} \Omega \Omega^{-1} x (x'\Omega^{-1}x)^{-1}$

$= \left[(x'x)^{-1} x' - (x'\Omega^{-1}x)^{-1} x'\Omega^{-1} \right] \Omega \left[x(x'x)^{-1} - \Omega^{-1} x (x'\Omega^{-1}x)^{-1} \right]$

$= \left[(x'x)^{-1} x' - (x'\Omega^{-1}x)^{-1} x'\Omega^{-1} \right] \Omega \left[(x'x)^{-1} x' - (x'\Omega^{-1}x)^{-1} x'\Omega^{-1} \right]'$

$= A \Omega A'$ which is pd as Ω is pd

2. Consider a scenario whereby we have two explanatory variables: one continuous (x_1) and one binary (x_2). Suppose we wish to model the relationship between the outcome variable (y) and the explanatory variables (x_1, x_2) via

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1^2 + \beta_4 x_1 x_2 + \beta_5 x_1^2 x_2 + e,$$

where $E(e|x_1, x_2) = 0$, $E(e^2|x_1, x_2) = \sigma^2(x_1, x_2) > 0$ where $\sigma^2(x_1, x_2)$ is a monotonically increasing function of x_1. Further, $\beta_0, \beta_1, \beta_2, \beta_4 > 0$ and $\beta_3, \beta_5 < 0$. With this information

(a) Write down the conditional expectation of y given x_1 and x_2 (for each value that x_2 can take).

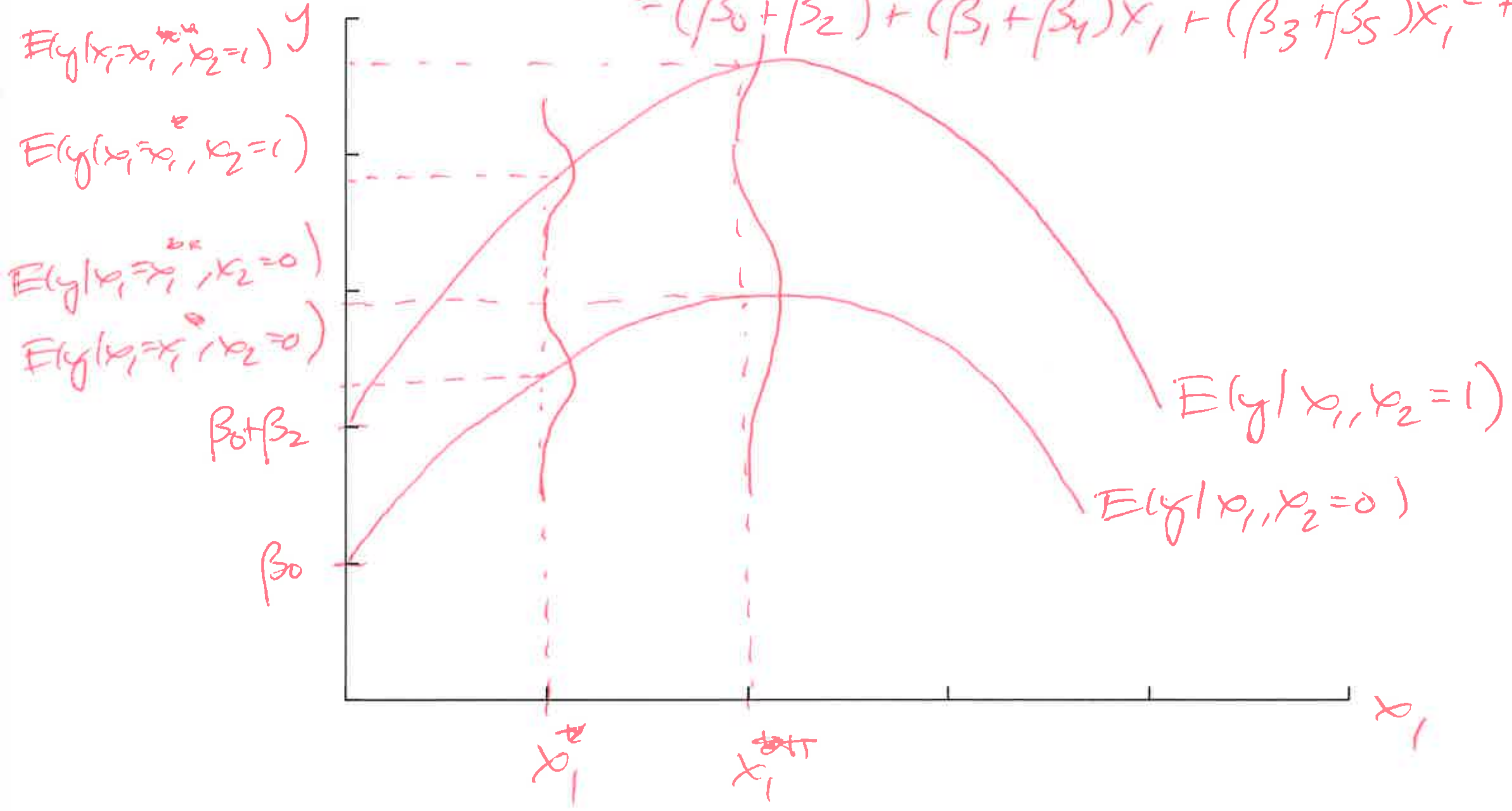
(b) In the figure below, draw the conditional expectations of y from part (a).

(c) Pick any two values for x_1, in the same figure, note the conditional expectation for each value (of x_1) you choose for each value x_2 can take. Plot and label a feasible distribution of the error (e) for each of the values for x_1.

(a) $E(y|x_1, x_2=0) = \beta_0 + \beta_1 x_1 + \beta_3 x_1^2 + e$

$E(y|x_1, x_2=1) = \beta_0 + \beta_1 x_1 + \beta_2 + \beta_3 x_1^2 + \beta_4 x_1 + \beta_5 x_1^2 + e$

$= (\beta_0 + \beta_2) + (\beta_1 + \beta_4) x_1 + (\beta_3 + \beta_5) x_1^2 + e$



error variance ↑ w/ x_1 $\quad \sigma^2(x_1^{**}, x_2) > \sigma^2(x_1^{*}, x_2)$

3. Consider the R code below. Next to each line of code, briefly comment on what that line of code is doing (write formulas if necessary).

```
## R code for Question 3 - MT1
```
comment

```
rm(list=ls())
```
clean memory

```
set.seed(10122021)
```
set random seed

```
n <- 1000
```
Sample size $= 1000$

```
x <- runif(n,0,1)
```
$x \sim U[0,1]$ from 0 to 1

```
e <- rnorm(n,0,1)
```
$e \sim N(0,1)$

```
y <- 1 + 0.5*x + x^2 + e
```
$y = \alpha + \beta x + \gamma x^2 + e = 1 + 0.5x + x^2 + e$

```
ones <- matrix(1,nrow=n,ncol=1)
```
column vector of ones

```
X <- cbind(ones,x,(x^2))
```
$X = (1 \; x \; x^2)$ data matrix

```
b <- solve(t(X)%*%X)%*%(t(X)%*%y)
```
OLS estimate $\hat{\beta} = (X'X)^{-1}X'y$

```
yh <- X%*%b
```
$\hat{y} = X\hat{\beta}$

```
eh <- y - yh
```
$\hat{e} = y - \hat{y}$

```
d <- diag(eh^2)
```
$V(e|X) = D = \mathrm{diag}(\hat{e}_1^2, \hat{e}_2^2, \ldots, \hat{e}_n^2)$

```
vb <- solve(t(X)%*%X)%*%t(X)%*%d%*%X%*%solve(t(X)%*%X)
```
$V(\hat{\beta}|X) = (X'X)^{-1}X'DX(X'X)^{-1}$