Economics 670: Econometrics

Department of Economics, Finance and Legal Studies

University of Alabama

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Final Exam Key

The exam consists of four questions on four pages. Each question is of equal value.

1. For $y = X\beta + e$, where X is an $n \times k$ matrix and β is a $k \times 1$ vector, consider the orthogonal projection matrix $M = I_n - P$, where I_n is an identity matrix of dimension n and $P = X(X'X)^{-1}X'$ is the projection matrix. With this information:

   (a) Show that M is symmetric.

   (b) Show that M is idempotent.

   (c) Show that the trace of M is $n - k$.

   (d) Show that $MX = 0$.

   (e) Show that $My = \widehat{e}$.

(a) $M' = (I_n - P)' = (I_n - X(X'X)^{-1}X')'$

$= I_n' - (X(X'X)^{-1}X')'$

$= I_n - X(X'X)^{-1}X' = I_n - P = M$

(b) $MM = (I_n - P)(I_n - P)$

$= I_nI_n - I_nP - PI_n + PP$

$= I_n - P - P + PP$

$= I_n - P - P + X(X'X)^{-1}X'X(X'X)^{-1}X'$

$= I_n - P - P + X(X'X)^{-1}X'$

$= I_n - P - P + P = I_n - P = M$

(c) $tr(M) = tr(I_n - P) = tr(I_n) - tr(P)$

$= n - tr[X(X'X)^{-1}X'] = n - tr[(X'X)^{-1}X'X]$

$= n - tr(I_k) = n - k$

(d) $MX = (I-P)X = (I - X(X'X)^{-1}X')X$

$= X - X(X'X)^{-1}X'X = X - X = 0$

(e) $My = (I-P)y = y - Py$

$= y - X(X'X)^{-1}X'y = y - X\hat{\beta} = \hat{e}$

2. Consider the model $y = X\beta + e$, where $E(e|X) = 0$ and $E(ee'|X) = \Omega$, where Ω is an unknown variance-covariance matrix.

   (a) Derive the OLS estimator $\widehat{\beta}$ of β.

   (b) Show that the estimator from part (a) is unbiased.

   (c) Using the result from part (a), propose an estimator $\widehat{\Omega}$ for the variance-covarance matrix Ω.

   (d) Using the result from part (c), and the objective function $\widetilde{e}'\widehat{\Omega}^{-1}\widetilde{e}$, derive the feasible generalized least squares estimator $\widetilde{\beta}$ of β.

   (e) Show that the estimator from part (d) is unbiased.

(a) $\hat{e}'\hat{e} = (y - x\hat{\beta})'(y - x\hat{\beta})$

$\frac{\partial}{\partial \beta} = x'(y - x\hat{\beta}) = 0$

$\Rightarrow x'y - x'x\hat{\beta} = 0 \Rightarrow \hat{\beta}_{OLS} = (x'x)^{-1}x'y$

(b) $\hat{\beta} = (x'x)^{-1}x'(x\beta + e) = \beta + (x'x)^{-1}x'e$

$E(\hat{\beta} - \beta | x) = (x'x)^{-1}x'E(e|x) = 0$

(c) $\hat{\Omega} = \hat{e}\hat{e}' = (y - x\hat{\beta})(y - x\hat{\beta})'$

(d) $\tilde{e}'\hat{\Omega}^{-1}\tilde{e} = (y - x\tilde{\beta})'\hat{\Omega}^{-1}(y - x\tilde{\beta})$

$\frac{\partial}{\partial \beta} = x'\hat{\Omega}^{-1}(y - x\tilde{\beta}) = 0$

$x'\hat{\Omega}^{-1}y - x'\hat{\Omega}^{-1}x\tilde{\beta} = 0$

$\Rightarrow \tilde{\beta}_{FGLS} = (x'\hat{\Omega}^{-1}x)^{-1}x'\hat{\Omega}^{-1}y$

(e) $\tilde{\beta}_{GLS} = (x'\hat{\Omega}^{-1}x)^{-1}x'\hat{\Omega}^{-1}(x\beta + e)$

$$= \beta + (x'\hat{\Omega}^{-1}x)^{-1}x'\hat{\Omega}^{-1}e$$

$$E(\tilde{\beta}_{GLS} - \beta | x) = (x'\hat{\Omega}^{-1}x)^{-1}x'\hat{\Omega}^{-1}E(e|x) = 0$$

3. Consider the model $y = X\beta + e$, where $e|X \sim N(0, \sigma^2 I_n)$. For an i.i.d. random sample of observations from this model,

(a) Derive the maximum likelihood estimator of β.

(b) Derive the variance of the estimator in part (a).

(c) Exploiting the assumption that $e|X \sim N(0, \sigma^2 I_n)$, show that the estimator from part (a) is asymptotically normal.

(d) Using large sample theory, show that the estimator from part (a) is consistent.

(e) Using large sample theory, show that the estimator from part (a) is asymptotically normal.

(a) $\ln \mathcal{L}(\beta, \sigma^2) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2}(y - x\beta)'(\sigma^2 I)^{-1}(y - x\beta)$

$= -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}e'e$

$\frac{\partial}{\partial\beta} = \frac{1}{\sigma^2}x'(y - x\beta) = 0$

$\Rightarrow x'y - x'x\beta = 0 \Rightarrow \hat{\beta} = (x'x)^{-1}x'y$

(b) $\frac{\partial^2}{\partial\beta\partial\beta'} = -\frac{1}{\sigma^2}x'x \Rightarrow \left[\cancel{E}\left(\frac{1}{\sigma^2}(x'x)\right)\right]^{-1}$

$= \sigma^2(x'x)^{-1}$

(c) $\hat{\beta} - \beta = (x'x)^{-1}x'e$

$\hat{\beta} - \beta | x \sim (x'x)^{-1}x'N(0, \sigma^2 I)$

$= N(0, (x'x)^{-1}x'\sigma^2 I x(x'x)^{-1})$

$= N(0, \sigma^2(x'x)^{-1})$

(d) $\hat{\beta} = \left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}X'y\right)$

WLLN $\begin{cases} \frac{1}{n}(X'X) \xrightarrow{p} E(x_i x_i') \\ \frac{1}{n}(X'y) \xrightarrow{p} E(x_i y_i) \end{cases}$

CMT $\hat{\beta} \xrightarrow{p} E(x_i x_i')^{-1} E(x_i y_i) = \beta$

(e) $\hat{\beta} - \beta = \left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}X'e\right)$

$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{\sqrt{n}}X'e\right)$

CLT $\frac{1}{\sqrt{n}}X'e \xrightarrow{d} N(0, \sigma^2 I)$

therefore

$\sqrt{n}(\hat{\beta} - \beta) \rightarrow N(0, \sigma^2 (X'X)^{-1})$

which is the same as $\hat{\beta}_{MLE}$ & shows that OLS is efficient under homoskedasticity

4. Consider the model $y = X\beta + e$, where β is a $k \times 1$ vector, $E(e|X) = 0$ and $V(e|X) = \sigma^2$. Suppose we are interested in the object $\theta = r(\beta) = 1_k' \times \beta = \beta_1 + \beta_2 + \cdots + \beta_k$.

   (a) What conditions are required to show that $\widehat{\theta}$ is a consistent estimator of θ?

   (b) What conditions are required to show that $\widehat{\theta}$ is asymptotically normal?

   (c) What is the variance of the distribution in part (b)? Be specific.

   (d) Suppose we wish to test the null $H_0 : \theta = r(\beta) = \beta_1 + \beta_2 + \cdots + \beta_k = 0$. Write down the Wald statistic for this null.

   (e) What is the asymptotic distribution of the test statistic from part (d)?

(a) need $\hat{\beta} \xrightarrow{p} \beta$ (finite variance, invertibility etc)
$r(\cdot)$ must be a continuous fn to invoke the CMT to show $\hat{\theta} \xrightarrow{p} \theta$

(b) in addition to that above, if the fn sufficiently smooth, the Delta method can show $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, V_\theta)$

(c) $V_\theta = R' V_\beta R$

$R = \frac{\partial}{\partial \beta} r(\beta)'$

$V_\beta = \sigma^2 (X'X)^{-1}$

$R = 1_k$

(d) $W = (\hat{\theta} - 0)' \hat{V}_\theta^{-1} (\hat{\theta} - 0)$

$= (\hat{\beta}_1 + \cdots + \hat{\beta}_k)' \hat{R}' \hat{V}_\beta^{-1} \hat{R} (\hat{\beta}_1 + \cdots + \hat{\beta}_k)$

$= (\hat{\beta}_1 + \cdots + \hat{\beta}_k)^2 / 1_k' \hat{\sigma}^2 (X'X)^{-1} 1_k \sim \chi^2_1$ (e)