# Economics 670: Econometrics

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Final -Key

The exam consists of four questions on five pages. Each question is of equal value.

1. Consider the model $y = \alpha + e$, for a random sample of $i = 1, 2, \ldots, n$ observations, where $e$ is normally distributed with mean 0 and variance $\sigma^2$. For this model,

   (a) Derive the method of moments estimator of $\alpha$.

   (b) Derive the ordinary least-squares estimator of $\alpha$.

   (c) Derive the maximum likelihood estimator of $\alpha$.

   (d) Suppose a researcher proposes another estimator of $\alpha$ as

   $$\widehat{\alpha} = \frac{n}{n+1}\widetilde{\alpha}$$

   where $\widetilde{\alpha}$ is your estimator from part (a), (b), or (c). Give the bias and variance of $\widehat{\alpha}$.

   (e) Show that $\widehat{\alpha}$ converges to a normal distribution and compare this with that of the estimator $\widetilde{\alpha}$.

(a) $E(e)=0 \Rightarrow \frac{1}{n}\sum_{i=1}^{n}(y_i-\hat{\alpha})=0 \Rightarrow \hat{\alpha}=\frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}$

(b) $\min_{\alpha} \hat{e}'\hat{e} \Rightarrow \frac{\partial}{\partial\alpha} = \frac{1}{n}\sum_{i=1}^{n}(y_i-\hat{\alpha}) \Rightarrow \hat{\alpha}=\frac{1}{n}\sum_{i=1}^{n} y_i=\bar{y}$

(c) $\ln\mathcal{L}(\theta) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\hat{\alpha})^2$

$\frac{\partial}{\partial\alpha} = \sum_{i=1}^{n}(y_i-\hat{\alpha}) \Rightarrow \hat{\alpha}=\frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}$

(d) $\hat{\alpha} = \frac{n}{n+1}\tilde{\alpha} = \frac{n}{n+1}\cdot\frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n+1}\sum_{i=1}^{n} y_i$

$\text{Bias}(\hat{\alpha}) = E(\hat{\alpha})-\alpha = \frac{n}{n+1}\alpha-\alpha = -\frac{1}{n+1}\alpha$ $(\to 0$ as $n\to\infty)$

$\text{Var}(\hat{\alpha}) = V\left(\frac{1}{n+1}\sum_{i=1}^{n} y_i\right) = \frac{n}{(n+1)^2}\sigma^2$ $(\to 0$ as $n\to\infty)$

(e) $V(\sqrt{n}(\hat{\alpha}-\alpha)) = \frac{n^2}{(n+1)^2}\sigma^2$ $(\to\sigma^2$ as $n\to\infty)$



$\sqrt{n}(\hat{\alpha}-\alpha) \xrightarrow{d} N(0,\sigma^2)$ as $n\to\infty$

which is asymptotically equivalent to the other

2. Consider the model $y = \alpha + e$, for a random sample of $i = 1, 2, \ldots, n$ observations, where $e$ is normally distributed with mean 0 and variance $\sigma^2$. Suppose we are interested in testing the null hypothesis $H_0 : \alpha = 1$. For this model and null:

(a) Construct the t-statistic and state its asymptotic distribution.

(b) Construct the Wald statistic and state its asymptotic distribution.

(c) Construct the likelihood ratio statistic and state its asymptotic distribution.

$$\hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} y_i = \bar{y} \qquad V(\hat{\alpha}) = \frac{\sigma^2}{n}$$

$$H_0: \alpha = 1$$

(a) $$t = \frac{\hat{\alpha} - 1}{\sigma / \sqrt{n}} \sim N(0,1)$$ (as $\sigma^2$ is known)

(b) $$W = (\hat{\alpha} - 1)' (V(\hat{\alpha}))^{-1} (\hat{\alpha} - 1)$$
$$= (\bar{y} - 1)' \left( \frac{\sigma^2}{n} \right)^{-1} (\bar{y} - 1)$$
$$= \frac{n(\bar{y} - 1)^2}{\sigma^2} \sim \chi^2_1$$

(c) $$LR = -2 \ln \lambda$$
$$= -2 (\ln \mathcal{L}(\alpha_0) - \ln \mathcal{L}(\alpha_1))$$
$$= -2 \Big( -\frac{n}{2} \ln 2\pi - \frac{n}{2} \ln \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - 1)^2$$
$$+ \frac{n}{2} \ln 2\pi + \frac{n}{2} \ln \sigma^2 + \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \bar{y})^2 \Big)$$
$$= \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - 1)^2 - \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \bar{y})^2$$
$$= \frac{n \sum_{i=1}^{n} (y_i - 1)^2}{\sigma^2} \sim \chi^2_1$$

3. Consider a random variable $Y$ which which follows a Bernoulli distribution

$$Y = \begin{cases} 1 \text{ with probability } \theta \\ 0 \text{ with probability } (1-\theta) \end{cases} \tag{1}$$

with probability density function

$$f_Y(y) = \theta^y (1-\theta)^{1-y}.$$

For this random variable:

(a) Derive the expected value of $Y$

(b) Derive the variance of $Y$

(c) What possible values can the parameter $\theta$ take?

(d) For a random sample of obserations of size $n$, derive the maximum likelihood estimator of $\theta$

(e) Suppose we wanted to test the null hypothesis that $H_0 : \theta = \theta_0$. Construct a test statistic for this null hypothesis and state its asymptotic distribution.

(a) $E(Y) = \theta(1) + (1-\theta)(0) = \theta$

(b) $E(Y^2) = \theta(1^2) + (1-\theta)(0^2) = \theta$

$V(Y) = E(Y^2) - E(Y)^2 = \theta - \theta^2 = \theta(1-\theta)$

(c) $0 \leq \theta \leq 1$ as $\theta$ is a probability

(d) $\ln \mathcal{L} = \sum_{i=1}^{n} y_i \ln \theta + \sum_{i=1}^{n} (1-y_i) \ln(1-\theta)$

$\frac{\partial}{\partial \theta} = \frac{1}{\theta} \sum_{i=1}^{n} y_i - \frac{1}{1-\theta} \sum_{i=1}^{n} (1-y_i) \Rightarrow$

$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} y_i$

(e) $H_0 : \theta = \theta_0$

$W = (\hat{\theta} - \theta)' (\hat{\theta}(1-\hat{\theta}))^{-1} (\hat{\theta} - \theta_0)$

$= \frac{n(\bar{y} - \theta_0)^2}{\bar{y}(1-\bar{y})} \sim \chi^2_1$

4. Considering the R code on the next page, answer the following:

   (a) What is the distribution of the error ($e$), regressor ($x$), and regressand ($y$)?

   (b) Consider the estimation method of the parameter, formally derive that estimator.

   (c) For the estimator in part (b), show what the weak law of large numbers implies.

   (d) What is the asymptotic distribution of the estimator in part (b).

   (e) If we were to repeat the test being performed in the model (say, a thousand times), what proportion of the time would we expect to reject the null as $n$ gets very large?

(a) $e \sim N(0,1)$, $x \sim N(0,1)$, $y = x + e \sim N(0,2)$

(b) $\min_{\beta} \sum_{i=1}^{n} \hat{e}_i^2 \Rightarrow \frac{\partial}{\partial \beta} = -2 \sum_{i=1}^{n} (y_i - \hat{\beta} x_i) x_i = 0$

$\Rightarrow \hat{\beta} = \sum_{i=1}^{n} y_i x_i \Big/ \sum_{i=1}^{n} x_i^2$

(c) WLLN $\left(\frac{1}{n} \sum_{i=1}^{n} y_i x_i\right) \xrightarrow{p} E(y_i x_i)$ $\Rightarrow \hat{\beta} \xrightarrow{p} \beta$

$\left(\frac{1}{n} \sum_{i=1}^{n} x_i^2\right) \xrightarrow{p} E(x_i^2)$

(d) $V(\hat{\beta}) = \dfrac{\sigma^2}{\sum_{i=1}^{n} (x_i)^2}$ $\qquad \frac{1}{n} \sum_{i=1}^{n} x_i^2 \xrightarrow{p} E(x_i^2)$

$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, V_\beta)$

$V_\beta = E(x_i^2)^{-1} \Omega E(x_i^2)^{-1} = \sigma^2 E(x_i^2)^{-1}$

(e) $H_0: \beta = 1$

$t = \dfrac{\hat{\beta} - 1}{se(\hat{\beta})}$

here the null is correctly specified & thus w/ expect the size (prob of rejection) to be equal to the signific level (say 5%)

```
n <- 100
b <- 1

x <- rnorm(n,0,1)
e <- rnorm(n,0,1)
y <- b*x + e

est <- lm(y~x-1)
bh <- est$coef

s2 <- (1/(n-1))*sum(est$res^2)
vb <- s2/sum((x-mean(x))^2)

th <- (bh-1)/sqrt(vb)
```