Economics 471: Introductory Econometrics

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Midterm I — Key

The exam consists of three questions on three pages. Each question is of equal value.

1. Consider the model $y_i = \beta x_i + u_i$ for $i = 1, 2, \ldots, n$. With this information, answer the following:

   (a) Write down the sample moment condition for the method of moments estimator.

   (b) Write down the objective function for the ordinary least-squares estimator.

   (c) Using the result from (a) *or* (b), derive the estimator of β.

   (d) Show that the estimator from part (c) is a consistent estimator of β.

   (e) Suppose the true model is $y_i = \alpha + \beta x_i + u_i$. Derive the bias of the estimator from part (c).

(a) $E(ux) = 0 \Rightarrow \frac{1}{n}\sum_{i=1}^{n} \hat{u}_i x_i = 0$

$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} (y_i - \hat{\beta} x_i) x_i = 0$

(b) $\min_{\beta} \sum_{i=1}^{n} \hat{u}_i^2 = \sum_{i=1}^{n} (y_i - \hat{\beta} x_i)^2$

(c) $\frac{1}{n}\sum_{i=1}^{n} (y_i - \hat{\beta} x_i) x_i = 0$

$\frac{1}{n}\sum_{i=1}^{n} y_i x_i - \hat{\beta}\frac{1}{n}\sum_{i=1}^{n} x_i^2 = 0$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2}$$

(d) $\hat{\beta} = \dfrac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2} = \dfrac{\sum_{i=1}^{n} (\beta x_i + u_i) x_i}{\sum_{i=1}^{n} x_i^2} = \dfrac{\beta \sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i^2} + \dfrac{\sum_{i=1}^{n} u_i x_i}{\sum_{i=1}^{n} x_i^2}$

$$= \beta + \frac{\sum_{i=1}^{n} u_i x_i}{\sum_{i=1}^{n} x_i^2}$$

$$E(\hat{\beta}|x) = E\left(\beta + \frac{\sum_{i=1}^{n} u_i x_i}{\sum_{i=1}^{n} x_i^2} \Big| x\right)$$

$$= \beta + \frac{1}{\sum_{i=1}^{n} x_i^2} \sum_{i=1}^{n} E(u_i x_i | x)$$

$$\overset{(}{=} \beta + \frac{1}{\sum_{i=1}^{n} x_i^2} \sum_{i=1}^{n} x_i \cancelto{0}{E(u_i|x)} \overset{(3)}{=} \beta$$

$$V(\hat{\beta}|x) = V\left(\beta + \frac{\sum_{i=1}^{n} u_i x_i}{\sum_{i=1}^{n} x_i^2} \Big| x\right)$$

$$\overset{\text{iid}}{=} \frac{1}{\left(\sum_{i=1}^{n} x_i^2\right)^2} \sum_{i=1}^{n} V(u_i x_i | x)$$

$$= \frac{1}{\left(\sum_{i=1}^{n} x_i^2\right)^2} \sum_{i=1}^{n} x_i^2 \cancelto{\sigma^2}{V(u_i|x)}$$

$$= \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2} \longrightarrow 0 \text{ as } n \to \infty$$

$\Rightarrow$ consistent

(c) $\hat{\beta} = \dfrac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2} = \dfrac{\sum_{i=1}^{n} (\alpha + \beta x_i + u_i) x_i}{\sum_{i=1}^{n} x_i^2}$

$$= \frac{\alpha \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2} + \frac{\beta \sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i^2} + \frac{\sum_{i=1}^{n} u_i x_i}{\sum_{i=1}^{n} x_i^2}$$

$$= \alpha \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2} + \beta + \frac{\sum_{i=1}^{n} u_i x_i}{\sum_{i=1}^{n} x_i^2}$$

$$E(\hat{\beta} \mid x) = E\left(\alpha \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2} + \beta + \frac{\sum_{i=1}^{n} u_i x_i}{\sum_{i=1}^{n} x_i^2} \,\middle|\, x \right)$$

$$\overset{(3)}{=} \alpha \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2} + \beta \neq \beta \text{ unless } \sum_{i=1}^{n} x_i = 0 \text{ or } \alpha = 0$$

$$\text{Bias} = \alpha \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2}$$

2. Consider the model $y_i = \alpha + \beta x_i + u_i$ for $i = 1, 2, \ldots, n$ and the corresponding residuals from that model $\widehat{u}_i = y_i - \widehat{y}_i = y_i - \widehat{\alpha} - \widehat{\beta} x_i$. With this information, answer the following:

   (a) Write down the Gauss-Markov Assumptions.

   (b) Using the Gauss-Markov Assumptions you wrote down in part (a), starting with the second equality sign, list which assumption(s) are being employed (if any) to move from one line to the next in the proof below.

$$
\begin{aligned}
E(\widehat{u}_i|x) &= E\left(y_i - \widehat{\alpha} - \widehat{\beta} x_i | x\right) \\
&= E\left[(\alpha + \beta x_i + u_i) - \widehat{\alpha} - \widehat{\beta} x_i | x\right] \quad \leftarrow (1) \\
&= \alpha + \beta E(x_i|x) + E(u_i|x) - E(\widehat{\alpha}|x) - E\left(\widehat{\beta} x_i|x\right) \\
&= \alpha + \beta x_i + E(u_i|x) - E(\widehat{\alpha}|x) - E\left(\widehat{\beta}|x\right) x_i \\
&= \alpha + \beta x_i + E(u_i|x) - \alpha - \beta x_i \quad \leftarrow (1-4) \\
&= E(u_i|x) \quad \leftarrow \\
&= 0 \quad \leftarrow (3)
\end{aligned}
$$

   (c) What does the proof in part (b) show?

   (d) Which assumption is not needed?

   (e) Why is the assumption in part (d) unnecessary?

(a) (1) $y = \alpha + \beta x + u$

(2) $E(u) = 0$

(3) $E(u|x) = 0$

(4) $\sum_{i=1}^{n} (x_i - \bar{x})^2 > 0$

(5) $V(u|x) = \sigma^2$

(c) conditional upon x, the residuals are zero in expectation

(d) homoskedasticity

(e) just unbiasedness, not variance

3. Consider the gretl output below on the regression of the log price (lprice) of a house on the log square footage (lsqrft) of the house. With this information, answer the following:

Model 1: OLS, using observations 1–88
Dependent variable: lprice

	Coefficient	Std. Error	t-ratio	p-value
const	−0.975	0.641	−1.521	0.132
lsqrft	0.872	0.084	10.310	0.000

Mean dependent var	5.633	S.D. dependent var	0.303
Sum squared resid	3.583	S.E. of regression	0.204
R^2	0.552	Adjusted R^2	0.547
$F(1, 86)$	106.389	P-value(F)	0.000
Log-likelihood	15.971	Akaike criterion	−27.943
Schwarz criterion	−22.988	Hannan–Quinn	−25.947

(a) Interpret the coefficient on lsqrft. Is this result intuitive?

(b) Test the null hypothesis that the slope is zero, i.e., $H_0 : \beta = 0$.

(c) What proprtion of the total variation in lprice can be explained by the model?

(d) Test the null hypothesis that the regression can be run through the origin.

(e) Using the conclusion from part (d), draw the regression (fitted) line on a graph.

(a) elasticity %Δ price / %Δ sqrft = 0.872

(b) $t_\beta = \frac{0.872 - 0}{0.084} = 10.310 > 2$ or p-value < 0.05

$\Rightarrow$ reject H_0

(c) $R^2 = 0.552$ or 55.2%

(d) $H_0: \alpha = 0$, $H_1: \alpha \neq 0$

$t_\alpha = \frac{-0.975 - 0}{0.641} = |-1.521| < 2$ or p-value > 0.05

$\Rightarrow$ fail to reject H_0

(e) lprice

$$\widehat{lprice} = \hat{\beta}\, lsqft$$