Thus, the covariance between y_t and y_{t-s} is constant and time-invariant for all t and t–s. Nothing of substance is changed by combining the AR(p) and MA(q) models into the general ARMA(p, q) model:

$$y_t = a_0 + \sum_{i=1}^{p} a_i\, y_{t-i} + x_t$$

$$x_t = \sum_{i=0}^{q} \beta_i \varepsilon_{t-i} \tag{2.22}$$

If the roots of the inverse characteristic equation lie outside of the unit circle [i.e., if the roots of the homogeneous form of (2.22) lie inside the unit circle] and if the $\{x_t\}$ sequence is stationary, the $\{y_t\}$ sequence will be stationary. Consider

$$y_t = \frac{a_0}{1-\sum_{i=1}^{p} a_i} + \frac{\varepsilon_t}{1-\sum_{i=1}^{p} a_i L^i} + \frac{\beta_1 \varepsilon_{t-1}}{1-\sum_{i=1}^{p} a_i L^i} + \frac{\beta_2 \varepsilon_{t-2}}{1-\sum_{i=1}^{p} a_i L^i} + \ldots \tag{2.23}$$

With very little effort, you can convince yourself that the $\{y_t\}$ sequence satisfies the three conditions for stationarity. Each of the expressions on the right-hand side of (2.23) is stationary as long as the roots of $1 - \Sigma a_i L^i$ are outside the unit circle. Given that $\{x_t\}$ is stationary, only the roots of the autoregressive portion of (2.22) determine whether the $\{y_t\}$ sequence is stationary.

5. THE AUTOCORRELATION FUNCTION

The autocovariances and autocorrelations of the type found in (2.18) serve as useful tools in the Box–Jenkins (1976) approach to identifying and estimating time-series models. We illustrate by considering four important examples: the AR(1), AR(2), MA(1), and ARMA(1, 1) models. For the AR(1) model, $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$, (2.14) shows

$$\gamma_0 = \sigma^2/[1-(a_1)^2]$$
$$\gamma_s = \sigma^2(a_1)^s/[1-(a_1)^2]$$

Forming the autocorrelations by dividing each γ_s by γ_0, we find that $\rho_0 = 1, \rho_1 = a_1$, $\rho_2 = (a_1)^2, \ldots, \rho_s = (a_1)^s$. For an AR(1) process, a necessary condition for stationarity is for $|a_1| < 1$. Thus, the plot of ρ_s against s—called the autocorrelation function (ACF) or **correlogram**—should converge to zero geometrically if the series is stationary. If a_1 is positive, convergence will be direct, and if a_1 is negative, the autocorrelations will follow a dampened oscillatory path around zero. The first two graphs on the left-hand side of Figure 2.2 show the theoretical autocorrelation functions for $a_1 = 0.7$ and $a_1 = -0.7$ respectively. Here, ρ_0 is not shown since its value is necessarily unity.

The Autocorrelation Function of an AR(2) Process

Now consider the more complicated AR(2) process $y_t = a_1 y_{t-1} + a_2 y_{t-2} + \varepsilon_t$. We omit an intercept term (a_0) since it has no effect on the ACF. For the second-order process to be stationary, we know that it is necessary to restrict the roots of $(1 - a_1L - a_2L^2)$ to be outside the unit circle. In Section 4, we derived the autocovariances of an ARMA(2, 1)

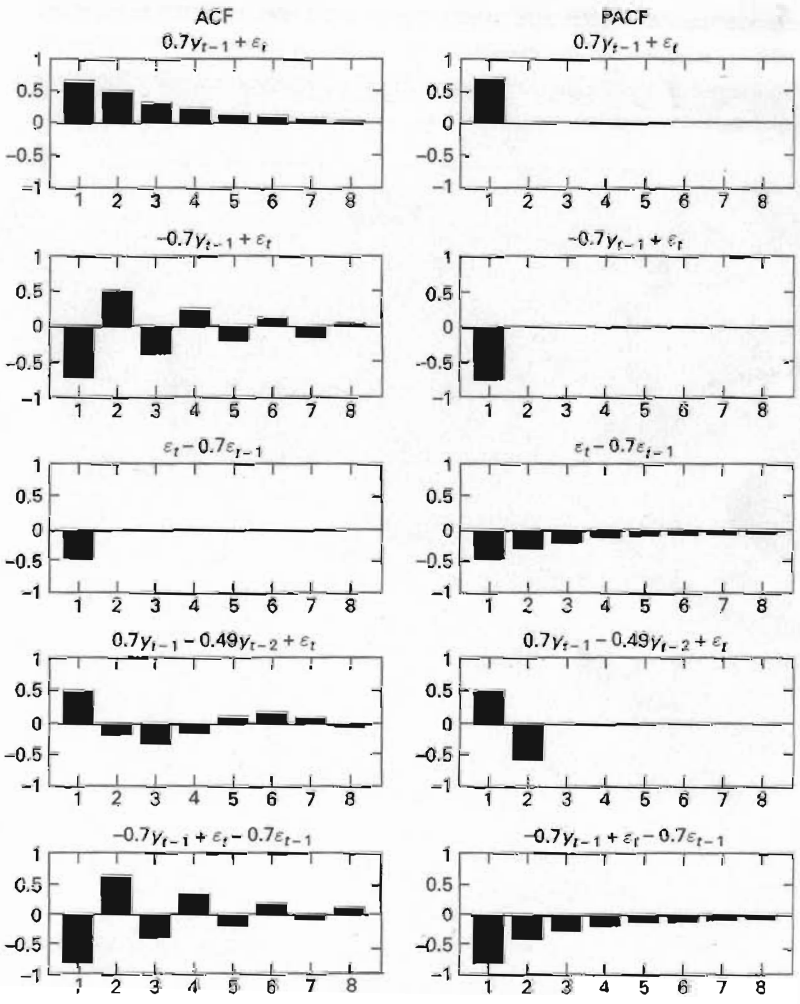

FIGURE 2.2 Theoretical ACF and PACF Patterns

process by use of the method of undetermined coefficients. Now we want to illustrate an alternative technique using the **Yule–Walker** equations. Multiply the second-order difference equation by y_{t-s} for $s = 0, s = 1, s = 2, \ldots$ and take expectations to form

$$Ey_t y_t = a_1 Ey_{t-1}y_t + a_2 Ey_{t-2}y_t + E\varepsilon_t y_t$$
$$Ey_t y_{t-1} = a_1 Ey_{t-1}y_{t-1} + a_2 Ey_{t-2}y_{t-1} + E\varepsilon_t y_{t-1}$$
$$Ey_t y_{t-2} = a_1 Ey_{t-1}y_{t-2} + a_2 Ey_{t-2}y_{t-2} + E\varepsilon_t y_{t-2}$$
$$\vdots$$
$$Ey_t y_{t-s} = a_1 Ey_{t-1}y_{t-s} + a_2 Ey_{t-2}y_{t-s} + E\varepsilon_t y_{t-s} \tag{2.24}$$