

Supplemental Material On

“Additive Nonparametric Regression in the Presence of Endogenous Regressors”

Deniz Ozabaci,¹ Daniel J. Henderson,² and Liangjun Su³

THIS APPENDIX PROVIDES PROOFS FOR SOME TECHNICAL LEMMAS IN THE ABOVE PAPER.

Proof of Lemma B.1. By straightforward moment calculations, we can show that $E\|Q_{n,PP} - Q_{PP}\|^2 = O(\kappa_1^2/n)$ under Assumption A1(i)-(ii) and A2(vi). Then (i) follows from Markov inequality. By Weyl inequality [e.g., Bernstein (2005, Theorem 8.4.11)] and the fact that $\lambda_{\max}(A) \leq \|A\|$ for any symmetric matrix A (as $|\lambda_{\max}(A)|^2 = \lambda_{\max}(AA) \leq \|A\|^2$), we have

$$\begin{aligned} \lambda_{\min}(Q_{n,PP}) &\leq \lambda_{\min}(Q_{PP}) + \lambda_{\max}(Q_{n,PP} - Q_{PP}) \\ &\leq \lambda_{\min}(Q_{PP}) + \|Q_{n,PP} - Q_{PP}\| = \lambda_{\min}(Q_{PP}) + o_P(1). \end{aligned}$$

Similarly,

$$\begin{aligned} \lambda_{\min}(Q_{n,PP}) &\geq \lambda_{\min}(Q_{PP}) + \lambda_{\min}(Q_{n,PP} - Q_{PP}) \\ &\geq \lambda_{\min}(Q_{PP}) - \|Q_{n,PP} - Q_{PP}\| = \lambda_{\min}(Q_{\kappa_1}) - o_P(1). \end{aligned}$$

Analogously, we can prove the second part of (ii). Thus (ii) follows. By the submultiplicative property of the spectral norm, (i)-(ii) and Assumption A2(i),

$$\begin{aligned} \|Q_{n,PP}^{-1} - Q_{PP}^{-1}\|_{\text{sp}} &= \|Q_{n,PP}^{-1}(Q_{PP} - Q_{n,PP})Q_{PP}^{-1}\|_{\text{sp}} \leq \|Q_{n,PP}^{-1}\|_{\text{sp}} \|Q_{PP} - Q_{n,PP}\|_{\text{sp}} \|Q_{PP}^{-1}\|_{\text{sp}} \\ &= O_P(1) O_P(\kappa_1/n^{1/2}) O_P(1) = O_P(\kappa_1/n^{1/2}), \end{aligned}$$

where we use the fact that $\|Q_{n,PP}^{-1}\|_{\text{sp}} = [\lambda_{\min}(Q_{n,PP})]^{-1} = [\lambda_{\min}(Q_{PP}) + o_P(1)]^{-1} = O_P(1)$ by (ii) and Assumption A2(i). Then (iii) follows. The proof of (iv)-(v) is analogous to that of (i)-(ii) and thus omitted. ■

Proof of Lemma B.2. (i) By Assumption A1(i) and A2(i), $E\|\xi_{nl}\|^2 = n^{-2}\text{tr}\{\sum_{i=1}^n E(P_i P_i' U_i^2)\} \leq n^{-1}(1 + d\kappa_1) \lambda_{\max}(Q_{PP, U_i}) = O(\kappa_1/n)$. Then $\|\xi_{nl}\|^2 = O_P(\kappa_1/n)$ by Markov inequality.

(ii) By the facts that $\|a\|_{\text{sp}}^2 = \|a\|^2$ for any vector a , $|a'b| \leq \|a\| \|b\|$ for any two conformable vectors a and b and that $\mathcal{A}'A\mathcal{A} \leq \lambda_{\max}(A) \|\mathcal{A}\|^2$ for any p.s.d. matrix A and conformable vector

¹Deniz Ozabaci, Dept of Economics, State Univ of New York, Binghamton, NY 13902-6000, (607) 777-2572, Fax: (607) 777-2681, e-mail: dozabac1@binghamton.edu.

²Daniel J. Henderson, Dept of Economics, Finance and Legal Studies, Univ of Alabama, Tuscaloosa, AL 35487-0224, (205) 348-8991, Fax: (205) 348-0186, e-mail: djhender@cba.ua.edu.

³Liangjun Su, School of Economics, Singapore Management University, 90 Stamford Road, Singapore, 178903; Tel: (65) 6828-0386; e-mail: ljsu@smu.edu.sg.

\varkappa , Cauchy-Schwarz inequality, Lemma B.1(ii) and Assumptions A2(iv), we have

$$\begin{aligned}
\|\zeta_{nl}\|^2 &= \|\zeta_{nl}\|_{\text{sp}}^2 = \lambda_{\max}(\zeta_{nl}\zeta_{nl}') \\
&= \max_{\|\varkappa\|=1} n^{-2} \sum_{i=1}^n \sum_{j=1}^n \varkappa' P_i P_j' \varkappa [m_l(\mathbf{Z}_i) - P_i' \boldsymbol{\alpha}_l] [m_l(\mathbf{Z}_j) - P_j' \boldsymbol{\alpha}_l] \\
&\leq \max_{\|\varkappa\|=1} \left\{ n^{-1} \sum_{i=1}^n \left\{ \varkappa' P_i P_i' \varkappa [m_l(\mathbf{Z}_i) - P_i' \boldsymbol{\alpha}_l]^2 \right\}^{1/2} \right\}^2 \\
&\leq O_P(\kappa_1^{-2\gamma}) \max_{\|\varkappa\|=1} \left\{ n^{-1} \sum_{i=1}^n \varkappa' P_i P_i' \varkappa \right\} \leq O_P(\kappa_1^{-2\gamma}) \lambda_{\max}(Q_{n,PP}) = O_P(\kappa_1^{-2\gamma}).
\end{aligned}$$

(iii) Noting that $X_{li} = m_l(\mathbf{Z}_i) + U_{li} = P_i' \boldsymbol{\alpha}_l + U_{li} + [m_l(\mathbf{Z}_i) - P_i' \boldsymbol{\alpha}_l]$, by Lemma B.1(ii), w.p.a.1 we have

$$\begin{aligned}
\tilde{\boldsymbol{\alpha}}_l - \boldsymbol{\alpha}_l &= \left(\sum_{i=1}^n P_i P_i' \right)^{-1} \sum_{i=1}^n P_i X_{li} - \boldsymbol{\alpha}_l \\
&= Q_{n,PP}^{-1} n^{-1} \sum_{i=1}^n P_i U_{li} + Q_{n,PP}^{-1} n^{-1} \sum_{i=1}^n P_i [m_l(\mathbf{Z}_i) - P_i' \boldsymbol{\alpha}_l] \\
&= Q_{n,PP}^{-1} \xi_{nl} + Q_{n,PP}^{-1} \zeta_{nl} \equiv a_{1l} + a_{2l}, \text{ say.}
\end{aligned} \tag{B.1}$$

Note that $a_{1l} = Q_{\kappa_1}^{-1} \xi_{nl} + r_{1nl}$ where $r_{1nl} = (Q_{n,PP}^{-1} - Q_{PP}^{-1}) \xi_{nl}$ satisfies that

$$\begin{aligned}
\|r_{1l}\| &\leq \left\{ \text{tr} \left[\left(Q_{n,PP}^{-1} - Q_{PP}^{-1} \right) \xi_{nl} \xi_{nl}' \left(Q_{n,PP}^{-1} - Q_{PP}^{-1} \right) \right] \right\}^{1/2} \\
&\leq \|\xi_{nl}\|_{\text{sp}} \left\| Q_{n,PP}^{-1} - Q_{PP}^{-1} \right\| = O_P(\kappa_1^{1/2}/n^{1/2}) O_P(\kappa_1^{1/2}/n^{1/2}) = O_P(\kappa_1/n)
\end{aligned}$$

by Lemmas B.1(iii) and B.2(i). For a_{2l} , we have $a_{2l} = Q_{\kappa_1}^{-1} \zeta_{nl} + r_{2nl}$ where $r_{2nl} = (Q_{n\kappa_1}^{-1} - Q_{n\kappa_1}^{-1}) \zeta_{nl}$ satisfies that

$$\|r_{2l}\| \leq \|\zeta_{nl}\|_{\text{sp}} \left\| Q_{n,PP}^{-1} - Q_{PP}^{-1} \right\| = O_P(\kappa_1^{-\gamma}) O_P(\kappa_1^{1/2}/n^{1/2}) = O_P(\kappa_1^{-\gamma+1/2}/n^{1/2})$$

by Lemmas B.1(iii) and B.2(ii). The result follows. ■

Proof of Lemma B.3. (i) We only prove the $r = 1$ case as the proof of the other case is almost identical. By the definition of \tilde{U}_{li} and (B.1), we can decompose $\tilde{U}_{li} - U_{li} = [X_{li} - \tilde{m}_l(\mathbf{Z}_i)] - U_{li}$

as follows

$$\begin{aligned}
\tilde{U}_{li} - U_{li} &= (\mu_l - \tilde{\mu}_l) + \sum_{k=1}^{d_1} [m_{l,k}(Z_{1k,i}) - \tilde{m}_{l,k}(Z_{1k,i})] + \sum_{k=1}^{d_2} [m_{l,d_1+k}(Z_{2k,i}) - \tilde{m}_{l,d_1+k}(Z_{2k,i})] \\
&= -(\tilde{\mu}_l - \mu_l) - \sum_{k=1}^{d_1} p^{\kappa_1}(Z_{1k,i})' \mathbb{S}_{1k} a_{1l} - \sum_{k=1}^{d_2} p^{\kappa_1}(Z_{2k,i})' \mathbb{S}_{1,d_1+k} a_{1l} \\
&\quad - \sum_{k=1}^{d_1} p^{\kappa_1}(Z_{1k,i})' \mathbb{S}_{1k} a_{2l} - \sum_{k=1}^{d_2} p^{\kappa_1}(Z_{2k,i})' \mathbb{S}_{1,d_1+k} a_{2l} \\
&\equiv -u_{1l,i} - u_{2l,i} - u_{3l,i} - u_{4l,i} - u_{5l,i}, \text{ say.} \tag{B.2}
\end{aligned}$$

Then by Cauchy-Schwarz inequality, $n^{-1} \sum_{i=1}^n (\tilde{U}_{li} - U_{li})^2 \sigma_i^2 \leq 5 \sum_{s=1}^5 n^{-1} \sum_{i=1}^n u_{sl,i}^2 \sigma_i^2 \equiv 5 \sum_{s=1}^5 V_{nl,s}$, say. Apparently, $V_{nl,1} = O_P(n^{-1})$ as $\tilde{\mu}_l - \mu_l = O_P(n^{-1/2})$.

$$\begin{aligned}
V_{nl,2} &= n^{-1} \sum_{i=1}^n \left(\sum_{k=1}^{d_1} p^{\kappa_1}(Z_{1k,i})' \mathbb{S}_{1k} a_{1l} \right)^2 \sigma_i^2 \\
&\leq d_1 \sum_{k=1}^{d_1} n^{-1} \sum_{i=1}^n (p^{\kappa_1}(Z_{1k,i})' \mathbb{S}_{1k} a_{1l})^2 \sigma_i^2 = d_1 \sum_{k=1}^{d_1} \text{tr}(\mathbb{S}_{1k} a_{1l} a_{1l}' \mathbb{S}_{1k}' Q_{n1k,pp}) \\
&\leq d_1 \sum_{k=1}^{d_1} \lambda_{\max}(Q_{n1k,pp}) \text{tr}(a_{1l} a_{1l}' \mathbb{S}_{1k}' \mathbb{S}_{1k}) \leq d_1 \sum_{k=1}^{d_1} \lambda_{\max}(Q_{n1k,pp}) \|\mathbb{S}_{1k}\|_{\text{sp}}^2 \|a_{1l}\|^2.
\end{aligned}$$

where $Q_{n1k,pp} = n^{-1} \sum_{i=1}^n p^{\kappa_1}(Z_{1k,i}) p^{\kappa_1}(Z_{1k,i})'$ such that $\lambda_{\max}(Q_{n1k,pp}) = O_P(1)$ by Assumption A3(ii) and arguments analogous to those used in the proof of Lemma B.1(ii). In addition, $\|\mathbb{S}_{1k}\|_{\text{sp}}^2 = \lambda_{\max}(\mathbb{S}_{1k} \mathbb{S}_{1k}') = 1$ and $\|a_{1l}\|^2 \leq \left\| Q_{n,PP}^{-1} \right\|_{\text{sp}}^2 \|\xi_{nl}\|^2 = O_P(1) O_P(\kappa_1/n) = O_P(\kappa_1/n)$ by Lemma B.1(iii) and B.2(i) and Assumption A2(i). It follows that $V_{nl,2} = O_P(1) \times 1 \times O_P(\kappa_1/n) = O_P(\kappa_1/n)$. Similarly, using the fact that $\|a_{2l}\|^2 \leq \left\| Q_{n,PP}^{-1} \right\|_{\text{sp}}^2 \|\zeta_{nl}\|^2 = O_P(1) O_P(\kappa_1^{-2\gamma})$, we have

$$\begin{aligned}
V_{nl,4} &= n^{-1} \sum_{i=1}^n \left(\sum_{k=1}^{d_1} p^{\kappa_1}(Z_{1k,i})' \mathbb{S}_{1k} a_{2l} \right)^2 \sigma_i^2 \leq d_1 \sum_{k=1}^{d_1} \lambda_{\max}(Q_{n1k,pp}) \|\mathbb{S}_{1k}\|_{\text{sp}}^2 \text{tr}(a_{2l} a_{2l}') \\
&= O_P(1) \times 1 \times O_P(\kappa_1^{-2\gamma}) = O_P(\kappa_1^{-2\gamma}).
\end{aligned}$$

By the same token, $V_{nl,3} = O_P(\kappa_1 n^{-1})$ and $V_{nl,5} = O_P(\kappa_1^{-2\gamma})$.

(ii) The result follows from (i) and the fact that $\max_{1 \leq i \leq n} \|\Phi_i\| = O_P(\varsigma_{0\kappa})$ under Assumption A2(vi).

(iii) By Assumption A2(vi), Taylor expansion and (i),

$$\begin{aligned}
n^{-1} \sum_{i=1}^n \left\| p^{\kappa}(\tilde{U}_{li}) - p^{\kappa}(U_{li}) \right\|^2 &= n^{-1} \sum_{i=1}^n \left\| \dot{p}^{\kappa}(U_{li}^{\dagger}) (\tilde{U}_{li} - U_{li}) \right\|^2 \\
&\leq O(\varsigma_{1\kappa}^2) n^{-1} \sum_{i=1}^n (\tilde{U}_{li} - U_{li})^2 = O_P(\varsigma_{1\kappa}^2 \nu_{1n}^2),
\end{aligned}$$

where U_{li}^\dagger lies between \tilde{U}_{li} and U_{li} .

(iv) By Assumption A2(vi), Taylor expansion and triangle inequality,

$$\left\| n^{-1} \sum_{i=1}^n \left[p^\kappa(\tilde{U}_{li}) - p^\kappa(U_{li}) \right] \Phi'_i \right\|_{\text{sp}}$$

is bounded by $\left\| n^{-1} \sum_{i=1}^n \dot{p}^\kappa(U_{li}) \Phi'_i (\tilde{U}_{li} - U_{li}) \right\|_{\text{sp}} + \frac{1}{2} \left\| n^{-1} \sum_{i=1}^n \ddot{p}^\kappa(U_{li}^\dagger) \Phi'_i (\tilde{U}_{li} - U_{li})^2 \right\|_{\text{sp}} \equiv T_{nl,1} + T_{nl,2}$, where U_{li}^\dagger lies between \tilde{U}_{li} and U_{li} . By triangle and Cauchy-Schwarz inequalities and (ii),

$$\begin{aligned} T_{nl,1} &\leq n^{-1} \sum_{i=1}^n \|\dot{p}^\kappa(U_{li})\|_{\text{sp}} \left\| \Phi'_i (\tilde{U}_{li} - U_{li}) \right\|_{\text{sp}} \\ &\leq \left\{ n^{-1} \sum_{i=1}^n \|\dot{p}^\kappa(U_{li})\|^2 \right\}^{1/2} \left\{ n^{-1} \sum_{i=1}^n \|\Phi_i\|^2 |\tilde{U}_{li} - U_{li}|^2 \right\}^{1/2} \\ &= O_P(\kappa^{1/2}) O_P(\varsigma_{0\kappa} \nu_{1n}) = O_P(\kappa^{1/2} \varsigma_{0\kappa} \nu_{1n}). \end{aligned}$$

By triangle inequality and (i), $T_{nl,2} \leq O(\varsigma_{0\kappa} \varsigma_{2\kappa}) n^{-1} \sum_{i=1}^n (\tilde{U}_{li} - U_{li})^2 = O_P(\varsigma_{0\kappa} \varsigma_{2\kappa} \nu_{1n}^2)$. Then (iv) follows.

(v) Let $\Gamma_{nl} \equiv [p^\kappa(\tilde{U}_{l1}) - p^\kappa(U_{l1}), \dots, [p^\kappa(\tilde{U}_{ln}) - p^\kappa(U_{ln})]]'$ and $\mathbf{e} = (e_1, \dots, e_n)'$. Then we can write $n^{-1} \sum_{i=1}^n [p^\kappa(\tilde{U}_{li}) - p^\kappa(U_{li})] e_i$ as $n^{-1} \Gamma'_{nl} \mathbf{e}$. Let $\mathbb{D}_n \equiv \{(\mathbf{X}_i, \mathbf{Z}_i, \mathbf{U}_i)\}_{i=1}^n$. By the law of iterated expectations, Taylor expansion, Assumptions A1(i), A3(ii) and A2(vi) and (i)

$$\begin{aligned} E \left\{ \left\| n^{-1} \Gamma'_{nl} \mathbf{e} \right\|^2 \mid \mathbb{D}_n \right\} &= n^{-2} E [\text{tr}(\Gamma'_n \mathbf{e} \mathbf{e}' \Gamma_n)] = n^{-2} E [\text{tr}(\Gamma'_n E(\mathbf{e} \mathbf{e}' \mid \mathbb{D}_n) \Gamma_n)] \\ &= n^{-2} \sum_{i=1}^n [p^\kappa(\tilde{U}_{li}) - p^\kappa(U_{li})]^2 \sigma_i^2 \\ &\leq O_P(\varsigma_{1\kappa}) n^{-2} \sum_{i=1}^n (\tilde{U}_{li} - U_{li})^2 \sigma_i^2 = O_P(n^{-1} \varsigma_{1\kappa}^2 \nu_{1n}^2). \end{aligned}$$

It follows that $\left\| n^{-1} \Gamma'_{nl} \mathbf{e} \right\| = O_P(n^{-1/2} \varsigma_{1\kappa} \nu_{1n})$ by the conditional Chebyshev inequality. ■

Proof of Lemma B.4. (i) Noting that $n^{-1} \sum_{i=1}^n \left\| \tilde{\Phi}_i - \Phi_i \right\|^2 = \sum_{l=1}^{d_x} n^{-1} \sum_{i=1}^n \left\| p^\kappa(\tilde{U}_{li}) - p^\kappa(U_{li}) \right\|^2$, the result follows from Lemma B.3(iii).

(ii) Noting that $\left\| n^{-1} \sum_{i=1}^n (\tilde{\Phi}_i - \Phi_i) \Phi'_i \right\|^2 = \sum_{l=1}^{d_x} \left\| n^{-1} \sum_{i=1}^n [p^\kappa(\tilde{U}_{li}) - p^\kappa(U_{li})] \Phi'_i \right\|^2$, the result follows from Lemma B.3(iv).

(iii) Noting that $\tilde{Q}_{n,\Phi} - Q_{n,\Phi} = n^{-1} \sum_{i=1}^n (\tilde{\Phi}_i \tilde{\Phi}'_i - \Phi_i \Phi'_i) = n^{-1} \sum_{i=1}^n (\tilde{\Phi}_i - \Phi_i)(\tilde{\Phi}_i - \Phi_i)' + n^{-1} \sum_{i=1}^n (\tilde{\Phi}_i - \Phi_i) \Phi'_i + n^{-1} \sum_{i=1}^n \Phi_i (\tilde{\Phi}_i - \Phi_i)'$, the result follows from (i)-(ii) and the triangle inequality.

(iv) By the triangle inequality $\left\| \tilde{Q}_{n,\Phi\Phi}^{-1} - Q_{\Phi\Phi}^{-1} \right\|_{\text{sp}} \leq \left\| \tilde{Q}_{n,\Phi\Phi}^{-1} - Q_{n,\Phi\Phi}^{-1} \right\|_{\text{sp}} + \left\| Q_{n,\Phi\Phi}^{-1} - Q_{\Phi\Phi}^{-1} \right\|_{\text{sp}}$.

Arguments like those used in the proof of Lemma B.1(ii) show that $\left\| \tilde{Q}_{n,\Phi\Phi}^{-1} \right\|_{\text{sp}} = \left[\lambda_{\min} \left(\tilde{Q}_{n,\Phi\Phi} \right) \right]^{-1} = \left[\lambda_{\min} \left(Q_{\Phi\Phi} \right) + o_P(1) \right]^{-1} = O_P(1)$ where the second equality follows from (iii) and Lemma B.1(ii). By the submultiplicative property of the spectral norm and (iii),

$$\begin{aligned} \left\| \tilde{Q}_{n,\Phi\Phi}^{-1} - Q_{n,\Phi\Phi}^{-1} \right\|_{\text{sp}} &= \left\| \tilde{Q}_{n,\Phi\Phi}^{-1} \left(\tilde{Q}_{n,\Phi\Phi} - Q_{n,\Phi\Phi} \right) Q_{n,\Phi\Phi}^{-1} \right\|_{\text{sp}} \\ &\leq \left\| \tilde{Q}_{n,\Phi\Phi}^{-1} \right\|_{\text{sp}} \left\| \tilde{Q}_{n,\Phi\Phi} - Q_{n,\Phi\Phi} \right\|_{\text{sp}} \left\| Q_{n,\Phi\Phi}^{-1} \right\|_{\text{sp}} \\ &= O_P \left(\kappa^{1/2} \varsigma_{0\kappa} \nu_{1n} + \varsigma_{0\kappa} \varsigma_{2\kappa} \nu_{1n}^2 \right). \end{aligned}$$

Similarly, $\left\| Q_{n,\Phi\Phi}^{-1} - Q_{\Phi\Phi}^{-1} \right\|_{\text{sp}} = O_P(\kappa/n^{1/2})$ by Lemma B.1(iii). It follows that $\left\| \tilde{Q}_{n,\Phi\Phi}^{-1} - Q_{\Phi\Phi}^{-1} \right\|_{\text{sp}} = O_P(\kappa^{1/2} \varsigma_{0\kappa} \nu_{1n} + \varsigma_{0\kappa} \varsigma_{2\kappa} \nu_{1n}^2)$.

(v) Noting that $\left\| n^{-1} \sum_{i=1}^n \left(\tilde{\Phi}_i - \Phi_i \right) e_i \right\|^2 = \sum_{l=1}^{d_x} \left\| n^{-1} \sum_{i=1}^n \left[p^\kappa \left(\tilde{U}_{li} \right) - p^\kappa \left(U_{li} \right) \right] e_i \right\|^2$, the result follows from Lemma B.3(v).

(vi) Let $\delta_i \equiv \bar{g}(X_i, Z_{1i}, U_i) - \Phi_i' \beta$. By triangle inequality, Assumption A2(v), Jensen inequality and (i), we have $\left\| n^{-1} \sum_{i=1}^n \left(\tilde{\Phi}_i - \Phi_i \right) \delta_i \right\| \leq O_P(\kappa^{-\gamma}) n^{-1} \sum_{i=1}^n \left\| \tilde{\Phi}_i - \Phi_i \right\| = O_P(\kappa^{-\gamma}) O_P(\varsigma_{1\kappa} \nu_{1n}) = O_P(\kappa^{-\gamma} \varsigma_{1\kappa} \nu_{1n})$. ■

Proof of Lemma B.5. The proof of (i)-(ii) is analogous to that of Lemma B.2 (i)-(ii), respectively. Noting that $\left\| Q_{\Phi\Phi}^{-1} \right\|_{\text{sp}} = O(1)$ by Assumption A2(ii), we can prove (iii) by showing that $\|T_{nl}\| = O_P(\nu_{1n})$, where $T_{nl} = n^{-1} \sum_{i=1}^n \Phi_i \delta_{li} (\tilde{U}_{li} - U_{li})$ where $\delta_{li} = \dot{p}^\kappa(U_{li})' \beta_{d_x+d_1+l}$. By triangle inequality and Assumptions A1(ii) and A2(iii) and (v)

$$\begin{aligned} c_{\delta_l} &\equiv \max_{1 \leq i \leq n} \|\delta_{li}\| \leq \sup_{u_l \in \mathcal{U}_l} \left\| \dot{g}_{d_x+d_1+l}(u_l) - \dot{p}^\kappa(u_l)' \beta_{d_x+d_1+l} \right\| + \sup_{u_l \in \mathcal{U}_l} \left\| \dot{g}_{d_x+d_1+l}(u_l) \right\| \\ &= O(\kappa^{-\gamma}) + O(1) = O(1). \end{aligned}$$

By (B.2), $T_{nl} = n^{-1} \sum_{i=1}^n \Phi_i \delta_{li} (\tilde{U}_{li} - U_{li}) = \sum_{s=1}^5 n^{-1} \sum_{i=1}^n \Phi_i \delta_{li} u_{sl,i} = \sum_{s=1}^5 T_{nl,s}$, say.

Let $\eta_{nlk} \equiv n^{-1} \sum_{i=1}^n \delta_{li} \Phi_i p^{\kappa_1}(Z_{1k,i})'$ and $\bar{\eta}_{lk} = E(\eta_{nlk})$. Then $\|\eta_{nlk} - \bar{\eta}_{lk}\| = O_P((\kappa \kappa_1/n)^{1/2})$ by Chebyshev inequality and

$$\|\bar{\eta}_{lk}\|_{\text{sp}}^2 = \left\| E \left[\delta_{li} \Phi_i p^{\kappa_1}(Z_{1k,i})' \right] \right\|_{\text{sp}}^2 \leq c_\delta^2 \lambda_{\max}(M) = O(1),$$

where $M \equiv E \left[\Phi_i p^{\kappa_1}(Z_{1k,i})' \right] E \left[p^{\kappa_1}(Z_{1k,i}) \Phi_i' \right]$ and we use the fact that M has bounded largest eigenvalue. To see the last point, first note that for $\kappa_1 \leq \kappa$, $E \left[\Phi_i p^{\kappa_1}(Z_{1k,i})' \right]$ is a submatrix of $A \equiv E(\Phi_i \Phi_i')$ which has bounded largest eigenvalue. Partition A as follows

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

where $A_{ij} = A'_{ji}$ for $i, j = 1, 2, 3$ and $E[\Phi_i p^{\kappa_1} (Z_{1k,i})'] = \begin{bmatrix} A'_{12} & A_{22} & A'_{32} \end{bmatrix}'$. Then

$$M = \begin{bmatrix} A_{12}A'_{12} & A_{12}A_{22} & A_{12}A'_{32} \\ A_{22}A'_{12} & A_{22}A_{22} & A_{22}A'_{32} \\ A_{32}A'_{12} & A_{32}A_{22} & A_{32}A'_{32} \end{bmatrix}.$$

By Thompson and Freede (1970, Theorem 2), $\lambda_{\max}(M) \leq \lambda_{\max}(A_{12}A'_{12}) + \lambda_{\max}(A_{22}A'_{22}) + \lambda_{\max}(A_{32}A'_{32})$. By Fact 8.9.3 in Bernstein (2005), the positive definiteness of A ensures that both $A_{12}A'_{12}$ and $A_{32}A'_{32}$ have finite maximum eigenvalues as both $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$ are also positive definite. In addition, $\lambda_{\max}(A_{22}A_{22}) = [\lambda_{\max}(A_{22})]^2$ is finite as A has bounded maximum eigenvalue. It follows that $\lambda_{\max}(M) = O(1)$. Consequently, $\|\eta_{nlk}\| = O_P(1 + (\kappa\kappa_1/n)^{1/2}) = O_P(1)$.

Analogously, noting that 1 is the first element of Φ_i , we can show that $\|n^{-1} \sum_{i=1}^n \Phi_i \delta_i\|_{\text{sp}} = O_P(1 + (\kappa/n)^{1/2}) = O_P(1)$. It follows that

$$\begin{aligned} \|T_{nl,1}\| &= \left\| n^{-1} \sum_{i=1}^n \Phi_i \delta_{li} \right\|_{\text{sp}} |\tilde{\mu}_l - \mu_l| = O_P(1) O_P(n^{-1/2}) = O_P(n^{-1/2}), \\ \|T_{nl,2} + T_{nl,4}\| &\leq \sum_{k=1}^{d_1} \|\eta_{nlk}\| \|\mathbb{S}_{1k}\|_{\text{sp}} (\|a_{1l}\| + \|a_{2l}\|) = O_P(1) O(1) O(\nu_{1n}) = O(\nu_{1n}), \end{aligned}$$

and $\|T_{nl,3} + T_{nl,5}\| = O(\nu_{1n})$ by the same token. Thus we have shown that $\|T_{nl}\| = O_P(\nu_{1n})$. ■

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